Interval Tournaments

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ABSTRACT

A directed graph is an interval digraph if to each vertex v there corresponds an ordered pair of intervals (S_v, T_v) such that $u \to v$ if and only if $S_u \cap T_v \neq \emptyset$. A tournament is an oriented complete graph. We characterize the tournaments that are interval digraphs via the existence of a large transitive subtournament and by forbidden subtournaments. A bipartite graph is an interval bigraph if to each vertex there corresponds an interval such that vertices are adjacent if and only if their corresponding intervals intersect and each vertex belongs to a different partite set. We capitalize on the equivalence of the models for interval digraphs and interval bigraphs and use results of Das, Roy, Sen, and West for interval digraphs, and results of Müller for interval bigraphs. © (Year) John Wiley & Sons, Inc.

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1. INTRODUCTION

We will generally use notation similar to that in [14], except we will use *arc* instead of edge for directed graphs. For example, in a directed graph (or digraph) D, if there is an arc *from* vertex u *to* vertex v, we denote this by $u \to v$, and the vertex set of D will be denoted $V(D)$. A digraph D is an *interval digraph* if to each vertex v there is an ordered pair of intervals of the real line (S_v, T_v) such that $u \to v$ if and only if $S_u \cap T_v \neq \emptyset$.

A *tournament* is an oriented complete graph, that is, for every pair of distinct vertices u and v, either $u \rightarrow v$ or $v \rightarrow u$ and never both, and $v \not\rightarrow v$ for every vertex v. We call a tournament with $n = |V(T)|$ an *n-tournament.* A *subtournament* of a tournament T is the tournament induced on a subset of the vertices of T; that is, if $U \subseteq V(T)$, then the subtournament induced on U is the tournament with $u \rightarrow v$ if $u, v \in U$ and $u \rightarrow v$ in T. The subtournament induced on a subset U of $V(T)$ with $|U| = k$ will be called a *k-subtournament.* We are interested in determining which tournaments are interval digraphs, and we will call a tournament that is an interval digraph an *interval tournament.* There are obvious restrictions we must impose on the collection of ordered pairs of intervals in order to represent a tournament. Namely, $S_u \cap T_u = \emptyset$, for each $u \in V(T)$, and $S_u \cap T_v \neq \emptyset$ or $S_v \cap T_u \neq \emptyset$ and not both. A digraph D is *transitive* if for any $x, y, z \in V(D)$, $x \rightarrow y$ and $y \rightarrow z$ imply $x \rightarrow z$. It is well-known that a tournament is transitive if and only if it has no 3-cycle, see [10]. Here is an example of an interval tournament: Let T be the 3-tournament that is a cycle on the vertices a, b, c. We can use the following as a representation for T. Let $(S_a, T_a) = ([0, 1], \{2.5\})$, $(S_b, T_b) = ([1, 2], \{0.5\})$, and $(S_b, T_b) = ([2, 3], \{1.5\})$.

Our two main results will (1) characterize the interval tournaments via forbidden subtournaments, and (2) show that a tournament is an interval tournament if and only if it has a transitive $(n-1)$-subtournament.

Interval digraphs were introduced in [5] and characterized in various ways; one such characterization, the one we will use here, involves adjacency matrices. The adjacency matrix of a digraph D will be denoted $A(D)$ and is the $(0, 1)$-matrix with a 1 in entry (i, j) if and only if $v_i \rightarrow v_j$. The following property characterizes the adjacency matrices of interval digraphs. A (0,1)-matrix has a *zero partition* if, after independent row and column permutations, each zero can be labeled R or C such that below every C is a C and to the right of every R is an R.

Theorem 1.1. *(Das, Roy, Sen, West, [5])* A directed graph D is an interval digraph if and only if $A(D)$ has a zero partition.

Interval digraphs and various subclasses of interval digraphs have been extensively studied, see [5], [11], [13], [7], and [12].

A bipartite graph with vertex partition $X \cup Y$ is an *interval bigraph* if to each vertex $v \in X \cup Y$ there corresponds an interval of the real line I_v such that vertices $x \in X$ and $y \in Y$ are adjacent if and only if $I_x \cap I_y \neq \emptyset$. If a graph B is bipartite with X and Y the partite sets, and E its edge set, we denote this by $B = (X, Y, E)$. Given a digraph D, we can build from it a bipartite graph $B(D)$ with $B(D) = (X, Y, E)$, $|X| = |Y| = |V(D)|$ where x_i and y_i correspond to v_i, giving $v_i \rightarrow v_j$ if and only if $x_i y_j \in E$. Observe that if D is an interval digraph, then $B(D)$ is an interval bigraph: simply let $I(x_i) = S(v_i)$ and $I(y_i) = T(v_i)$. So, essentially, the models for interval digraphs and interval bigraphs are the same, but each perspective, bipartite graph or digraph, has led to a distinct collection of results. Interval bigraphs, for example, have a rich recent history, and, asking the reader to consult the references for definitions not found herein, we discuss a few of the recent results. The most significant of the recent results are those of P. Hell and J. Huang in [6]. In this paper, interval bigraphs and proper interval bigraphs are characterized in various ways, in particular, the complements of each of these classes are shown to be precisely a certain subclass of 2-clique circular arc graphs. Interval-point bigraphs are shown to be, among other things, precisely a subclass of bipartite probe interval graphs, and the complements of a subclass of 2-clique circular arc graphs in [1]. However, although unit interval bigraphs are completely characterized by forbidden induced subgraphs, see [3] and [6], the only class of interval bigraphs in general that are completely characterized by forbidden induced subgraphs are the interval bigraphs that are trees, see [4]. In fact, the results in [6] indicate the difficulty one will face in characterizing interval bigraphs by forbidden induced subgraphs. There are no forbidden subdigraph characterizations for interval digraphs, except the ones that extend from those for interval bigraphs. One of our main results here will be a characterization for interval tournaments via a list of forbidden subtournaments. See [2] for a survey of the recent history related to interval digraphs and interval bigraphs, going back to [5], the seminal paper that spurred a lot of this research.

2. CHARACTERIZATION

In order to characterize interval tournaments, we will capitalize upon the fact that if D is an interval digraph, then $B(D)$ is an interval bigraph. We will also use the fact that any subtournament of an interval tournament is an interval tournament; that is, being an interval tournament is a hereditary property. Some of the recent results on interval bigraphs indicate structural limitations that we can use. Specifically, for a tournament T, if $B(T)$ is not an interval bigraph on account of it having a forbidden substructure for interval bigraphs, then we can conclude T is not an interval tournament. Theorem 2.1 lists the results about interval bigraphs that we will use. An *asteroidal triple of edges* is a set of three edges in a bipartite graph with a path between any two edges that avoids the neighborhood of the third, where the neighborhood of an edge is the union of the neighborhoods of its end-vertices.

Theorem 2.1. *(Müller, [9])* If B is an interval bigraph, then the following hold:
(1) B has no induced cycle on 5 or more vertices;
(2) B has no induced subgraph isomorphic to the graph in Figure 1;
(3) B cannot have an asteroidal triple of edges in any induced subgraph.

When we draw a tournament, we show only the arcs directed to the right; all other arcs not shown are directed to the left. The graph in Figure 1 is $B(T_4)$, where T_4 is depicted in Figure 2. Therefore, no tournament with T_4 as a subtournament is an interval digraph. A digraph is *strong* if between any two vertices there is a directed path from one to the other. If a digraph is not strong, then its vertices can be partitioned into sets on which there are subdigraphs called *strong components*; these are induced subdigraphs that are

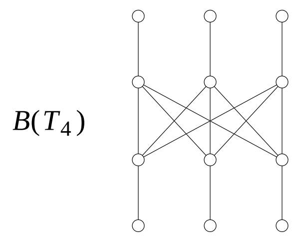


FIGURE 1. A bipartite graph that is not an interval bigraph.

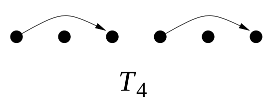


FIGURE 2. T_4, a 6-tournament that is not an interval tournament.

maximal with respect to being strong. Since a tournament cannot have both $u \to v$ and $v \to u$, and $v \to v$ never happens, a nontrivial strong component in a tournament has at least three vertices, and if a tournament has a cycle of any length, it has a 3-cycle. Also, for any digraph, the arcs between two strong components A and B must all go either from A to B or vice-versa. We remind the reader that interval tournaments have the property that any subtournament of an interval tournament must be an interval tournament. Theorem 2.1 (2) and these facts give Corollary 2.1.

Corollary 2.1. An interval tournament can have at most one nontrivial strong component.

Proof. Suppose T has two nontrivial strong components. Then each nontrivial strong component has a 3-cycle, and hence T contains T_4 as a subtournament. ∎

We also use Theorem 2.1 to show that T_1, T_2, and T_3 of Figure 3 are not interval tournaments. Figure 4 lists the bipartite graphs obtained from T_1, T_2, and T_3; since each has a cycle of length 6 or greater, the corresponding tournaments are not interval tournaments. In fact T_1, T_2, and T_3 are, at once, the only 5-tournaments that are not interval tournaments, and the only ones with no transitive 4-subtournament. Furthermore, it turns out that all 6-tournaments that are interval tournaments have a transitive 5-subtournament. These observations suggest the possibility that an n-tournament T having a transitive $(n-1)$-subtournament is a sufficient condition for T to be an interval tournament. In Lemma 2.1 we show that this is in fact the case.

Lemma 2.1. If an n-tournament T has a transitive $(n-1)$-subtournament, then T is an interval tournament.

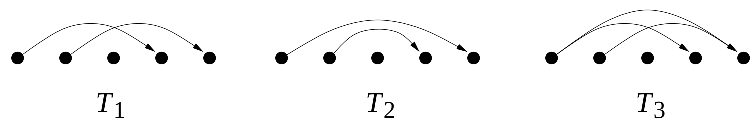


FIGURE 3. The three 5-tournaments that are not interval tournaments.

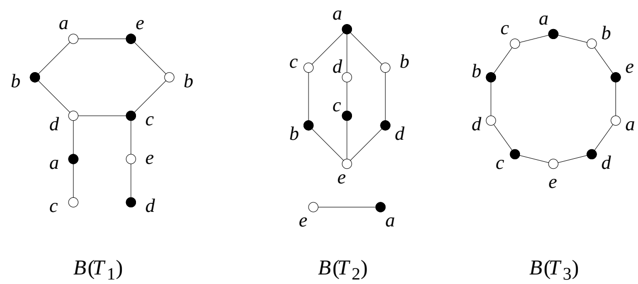


FIGURE 4. The bipartite graphs derived from T_1, T_2, and T_3.

Proof. We will show that the adjacency matrix of an n-tournament with a transitive $(n-1)$-subtournament has a zero partition. Suppose $T-v$ is transitive, and that the vertices of $T-v$ are labeled from 1 to $n-1$ so that $v_i \to v_j$ if and only if $i < j$. Let the columns and the rows of $A(T-v)$ correspond, in order, to $v_{n-1}, v_{n-2}, \ldots, v_1$. This matrix is lower triangular; we may label all the zeros R. Now append a column to this matrix for v, putting it first, and a row for v, putting it last. Permute rows of this matrix so that all 1's in the first column (for v) are on top; this does not require us to change the labeling given to the zeros. Now, all the zeros in the first column and last row (the column and row for v) can be labeled C. Thus, $A(T)$ has a zero partition, and T is an interval tournament, by Theorem 1.1. ∎

Case by case consideration shows that the converse of Lemma 2.1 is true for $n \leq 6$. We observe that T_4 has no transitive 5-subtournament, since any 5-subtournament must contain either the left-hand or right-hand 3-cycle. In a similar way, any 6-tournament with two vertex-disjoint 3-cycles has no transitive 5-subtournament. So, we consider all nonisomorphic 6-tournaments that have two vertex disjoint 3-cycles. Note, any tournament with two vertex disjoint 3-cycles must contain a subtournament isomorphic to one of these 6-tournaments. Thus, if we can show that none of these 6-tournaments are interval tournaments, it follows that no interval tournament contains two disjoint three cycles.

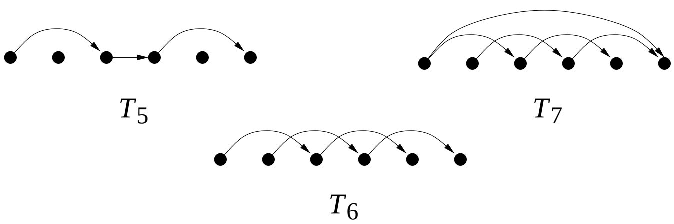


FIGURE 5. 6-tournaments with no induced T_1, T_2 or T_3 that are not interval tournaments.

All nonisomorphic 6-tournaments, modulo one error[1], can be found in an appendix of [8]. There are twenty five 6-tournaments with two vertex-disjoint 3-cycles, four of which are T_4 (Figure 2), T_5, T_6 and T_7 (Figure 5). The other 21 are listed in Table I, in which we indicate how to find, in each of these 21, T_1, T_2 or T_3 as a subtournament. Thus, none of the 21 6-tournaments in Table I is an interval tournament, by our observations about T_1, T_2, and T_3 above. Let us consider T_4, T_5, T_6, and T_7. The two vertex disjoint 3-cycles in each are easy to find. And we have shown that T_4 is not an interval tournament. For the tournaments T_5, T_6, and T_7, we consider their corresponding bipartite graphs, and then see that Theorem 2.1 shows that they are not interval tournaments. Figure 6 lists the bipartite graphs $B(T_5), B(T_6)$, and $B(T_7)$, corresponding to T_5, T_6, and T_7, respectively. Each of the first two has an asteroidal triple of edges on the bold edges, and the third has an induced 6-cycle. We can now conclude that no 6-tournament, and hence no n-tournament, with two vertex-disjoint 3-cycles is an interval tournament. We record this fact as Lemma 2.2; and we record the observations we have made regarding T_i, for $i = 1, 2, \ldots, 7$ as Lemma 2.3.

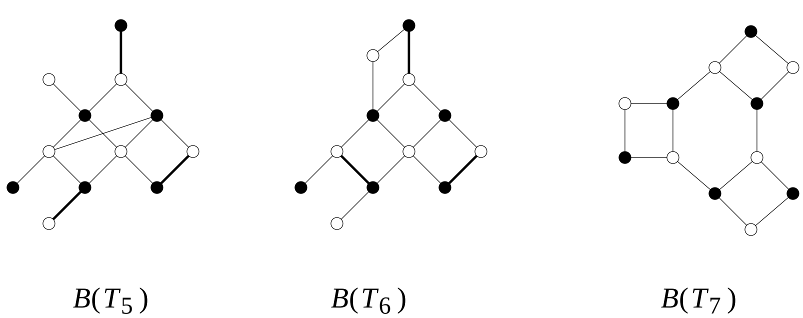


FIGURE 6. The bipartite graphs derived from T_5, T_6, and T_7. The bold edges indicate ATEs.

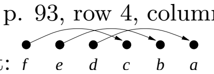
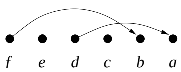
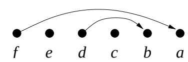
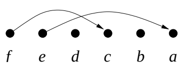
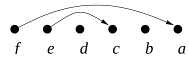
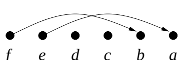
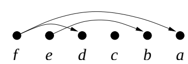
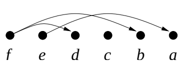
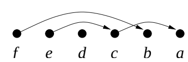
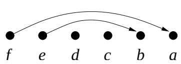
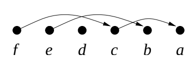
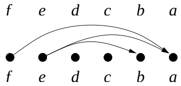
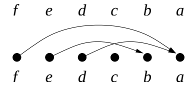
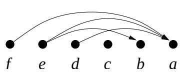
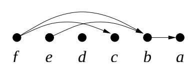
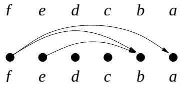
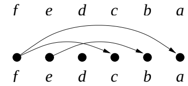
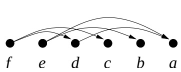
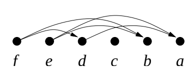
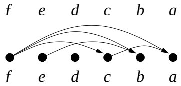
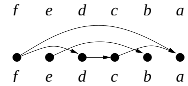
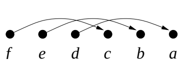
[1]The error consists of listing the tournament on p. 93, row 4, column 2, again on on p. 95, row 1, column 1, and the absence of this tournament: 

TABLE I. The 6-tournaments with vertex-disjoint 3-cycles not isomorphic to T_4, T_5, T_6, or T_7.

Tournament: T	3-cycle	Subtournament	Tournament: T	3-cycle	Subtournament
	$\{a, c, d\}$	$T - e \cong T_1$		$\{a, e, f\}$	$T - e \cong T_2$
	$\{a, b, e\}$	$T - b \cong T_1$		$\{a, b, f\}$	$T - b \cong T_2$
	$\{a, d, e\}$	$T - d \cong T_1$		$\{a, c, f\}$	$T - c \cong T_1$
	$\{a, d, e\}$	$T - d \cong T_1$		$\{a, e, c\}$	$T - a \cong T_2$
	$\{a, c, f\}$	$T - c \cong T_2$		$\{a, f, c\}$	$T - a \cong T_1$
	$\{a, c, f\}$	$T - c \cong T_1$		$\{a, c, f\}$	$T - c \cong T_1$
	$\{a, c, f\}$	$T - f \cong T_3$		$\{a, e, b\}$	$T - a \cong T_3$
	$\{a, c, f\}$	$T - d \cong T_1$		$\{a, d, f\}$	$T - c \cong T_2$
	$\{a, f, d\}$	$T - f \cong T_3$		$\{a, f, d\}$	$T - f \cong T_3$
	$\{a, d, f\}$	$T - a \cong T_3$		$\{a, e, f\}$	$T - f \cong T_1$
	$\{a, b, d\}$	$T - a \cong T_1$			

Lemma 2.2. If a tournament has two vertex-disjoint 3-cycles, it is not an interval tournament.

Lemma 2.3. If a tournament T has T_1, T_2, T_3 (of Figure 3), T_4 (of Figure 2), T_5, T_6, or T_7 (of Figure 5) as a subtournament, then T is not an interval tournament.

Besides the 21 tournaments listed in Table I and T_4, T_5, T_6, and T_7, there are ten other 6-tournaments that are not interval tournaments. Investigation of these shows that they contain T_1, T_2, or T_3 as a subtournament, and hence contain no transitive 5-tournament. Furthermore, examining all tournaments on $n = 6$ or fewer vertices, we see that every non-interval tournament has two vertex-disjoint 3-cycles or has T_1, T_2, or T_3 as a subtournament. These observations lead to the next result that shows, for any n-tournament T, there are two vertex-disjoint 3-cycles in T, a transitive $(n-1)$-subtournament, or T_1, T_2, or T_3 as a subtournament. We remind the reader of some notation we will use. Let v be a vertex of tournament T. The *out-degree of* v is the size of the set $N^+(v) = \{u : v \rightarrow u \text{ in } T\}$, and is denoted $d^+(v)$; the minimum out-degree of T is denoted $\delta^+(T)$. Similarly $d^-(v)$ denotes the *in-degree of* v and is the number of vertices u for which $u \rightarrow v$; the minimum in-degree of T is denoted $\delta^-(T)$.

Theorem 2.2. Let T be an n-tournament with no two vertex-disjoint 3-cycles. Then T has either a transitive $(n-1)$-subtournament, or a subtournament isomorphic to T_1, T_2, or T_3.

Proof. Let T be an n-tournament; the proof is by induction on n. The result holds for all tournaments on fewer than five vertices. No 5-tournament has two vertex disjoint 3-cycles, and the only ones that do not contain transitive 4-subtournaments are T_1, T_2, and T_3. We assume the result holds for $n-1$ and work with $n \geq 6$. We claim it suffices to show the result for strong tournaments. Suppose T is not strong. If T has two nontrivial strong components, it has at least two vertex-disjoint 3-cycles. If T has only one nontrivial strong component, call it T^* with k vertices, then by the induction hypothesis, T^* has T_1, T_2, or T_3 as a subtournament, and so does T, or T^* has a transitive $(k-1)$-subtournament. The latter case implies T has a transitive $(n-1)$-subtournament when the transitive $(k-1)$-subtournament in T^* is taken with the other trivial components.

Let T be a strong n-tournament with no two vertex-disjoint 3-cycles. Observe that the following tournaments are unchanged upon reversal of each arc: T_1, T_2, T_3, a transitive tournament, and a 3-cycle. Hence we may assume that $\delta^-(T) \geq \delta^+(T)$. Define $tr(T)$ to be the number of vertices on which the largest transitive subtournament exists in T. If $tr(T) < n-2$, then $T - v$, for any vertex v, is not transitive and $T - v$ has no two vertex-disjoint 3-cycles, since T doesn't. Any transitive subtournament of $T - v$ is a transitive subtournament in T, so $tr(T - v) \leq tr(T) < n - 2$. Hence the induction hypothesis applied to $T - v$ gives T_1, T_2, or T_3 as a subtournament of $T - v$ and also of T. Thus, we may assume that $tr(T) = n - 2$; so there is a vertex of degree at most two, and since T is strong there is no vertex with out-degree zero. We will consider the two cases $\delta^+(T) = 2$, and $\delta^+(T) = 1$.

Case 1: $\delta^+(T) = 2$. In this case there are vertices u and v such that $T - \{u, v\}$ is transitive. Assume without loss of generality that $u \rightarrow v$. Let x and z be the source and sink of $T - \{u, v\}$, respectively. Since $\delta^+(T) = 2$, we must have $z \rightarrow u$ and $z \rightarrow v$ in T, and since $\delta^-(T) \geq \delta^+(T)$, we must have $u \rightarrow x$ and $v \rightarrow x$. The degree conditions on T also require that there is some vertex y with $y \rightarrow u$ and some vertex y' with $v \rightarrow y'$. But $\{x, y, u\}$ and $\{v, y', z\}$ are both 3-cycles of T and hence cannot be vertex-disjoint; so, $y = y'$. Thus the subtournament on $\{x, y = y', z, u, v\}$ is a regular 5-tournament and so is isomorphic to T_3.

Case 2: $\delta^+(T) = 1$. Let u be a vertex with $d^+(u) = 1$ and let $N^+(u) = \{v\}$. Without loss of generality, assume that $d^+(v) > 1$; if this is not the case, then $N^+(v) = \{w\}$ and $d^+(w) > 1$ since no strong tournament on at least 5 vertices can have 3 vertices of out-degree 1. Now, consider $T - v$. No cycle of $T - v$ contains u, since u has out-degree zero in $T - v$. Furthermore this tournament is not transitive, by assumption, and so there must be a 3-cycle. So, suppose the vertices x, y, and z are a 3-cycle in $T - v$. Any vertex

$w \in N^+(v)$ is part of a 3-cycle of the form $v \to w \to u \to v$. But such a 3-cycle cannot be vertex-disjoint from the one on x, y, and z, so $N^+(v) \subseteq \{x, y, z\}$. Since $d^+(v) > 1$, either $d^+(v) = 2$, or $N^+(v) = \{x, y, z\}$. In the former case the tournament on $\{u, v, x, y, x\}$ is isomorphic to T_1, and in the latter case the tournament on $\{u, v, x, y, z\}$ is isomorphic to T_2. ∎

Putting Lemmas 2.2 and 2.3 together with Theorem 2.2, we get the following corollary.

Corollary 2.2. A tournament with no transitive $(n-1)$-subtournament is not an interval tournament.

We now have what we need for a characterization of interval tournaments. Lemma 2.1 and Corollary 2.2 give the characterization for interval tournaments in Theorem 2.3 (2). Lemmas 2.3 and 2.2 and Theorem 2.2 give the characterization in Theorem 2.3 (3).

Theorem 2.3. Let T be an n-tournament. The following are equivalent.
(1) T is an interval tournament;
(2) T has a transitive $(n-1)$-subtournament;
(3) T has no subtournament isomorphic to $T_1, T_2, T_3, T_4, T_5, T_6$, or T_7.

3. TWO QUESTIONS

In [6] P. Hell and J. Huang show that interval bigraphs are equivalent to a very specific class of 2-clique circular arc graphs. Other forbidden substructures, different from the ones in Theorem 2.1 are also given. But they indicate that there are other bipartite graphs that are not interval bigraphs yet free of any of the known substructures that are forbidden for interval bigraphs. We believe that two questions follow from these results of Hell and Huang and by some of the observations the authors have made during the investigations for this paper. One is whether the interval tournaments can be characterized by a concise description of the 2-clique circular arc graphs that arise from their complements. The other question is whether adopting the digraph perspective, and attempting to characterize other subclasses of interval digraphs, will lead to more forbidden substructures for interval bigraphs, or will it suggest a way to describe the ones that do not fit into the categories described in [9] and [6].

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