Extremal Results on Arc-traceable Tournaments

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Abstract

A tournament $T = (V, A)$ is *arc-traceable* if for each arc xy in A, xy lies on a directed path containing all the vertices of V, i.e. a hamiltonian path. In this paper we give two extremal results related to arc-traceability in tournaments. First, we show that a non-arc-traceable tournament T which is m arc-strong must have at least $2^{m+1} + 4m - 3$ vertices, and we construct an example that shows this result is best possible. Next, we consider the maximum number of arcs in a strong tournament that are not part of any hamiltonian path. There exists a strong tournament containing $\frac{n^2-4n+3}{8}$ such arcs, and in this paper we use the structure of non-arc-traceable tournaments to prove that no strong tournament contains more than this number of arcs that are not part of a hamiltonian path.

1 Introduction

State results from Quintas paper and structure paper.

Theorem 1.1 (Busch, Jacobson and Reid [3]). *A strong tournament that is not arc-traceable has a given structure.*

Figure

Note that upset tournaments are 1-arc-strong, but that by using the same ideas, we will construct k-arc-strong non-arc-traceable tournaments.

Note the upset tournament that gives the maximum number of non-traceable arcs (among all upset tournaments) from the Quintas paper. By extending the results from the structure paper, we will show that this example is in fact best possible for all tournaments.

2 Non-arc-traceable k-arc-strong tournaments

In [3], it was shown that for a non-traceable arc xy in a strong tournament T, there exists a y,x separating set of vertices with size 1. We begin by showing that a similar result for a separating set of arcs is impossible. In other words, for any $m > 0$ there exists a strong tournament T that has a non-traceable arc xy with m arc-disjoint paths from y to x. As an example, in the tournament in Figure 2 xy is not on any hamiltonian path, and there exist 2 arc-disjoint paths from y to x.

For an arbitrary m, we construct a strong tournament by reversing the arcs of a set of m arc-disjoint paths in a transitive $(2^{m+1}+1)$-tournament. Let $n = 2^{m+1}+1$, and let T be a transitive n-tournament with vertices labeled $v_0, \ldots, v_{2^m+1}$ such that $d^+(v_i) = i$. Now consider the paths

$$P_i = v_{2^{m+1}} v_{(2^{m+1}-2^i)} v_{(2^{m+1}-2 \cdot 2^i)} v_{(2^{m+1}-3 \cdot 2^i)} \ldots, v_0, \text{ for } 1 \leq i \leq m.$$

We reverse the arcs in each of these paths to obtain the tournament $T_{[m]}$, and will refer to the reversed path P_i as U_i. Note that v_{2^m} is on each U_i, and thus for any $i < 2^m < j$, v_m separates v_i and v_j. Also, observe that we can view the construction of $T_{[m+1]}$ recursively; take two copies of $T_{[m]}$ sharing a single vertex (the terminal endpoint of all the upset paths from one copy, and the initial endpoint of the upset paths in the other copy) and reverse the 2-path P_{m+1}. This recursive perspective allows for the easy application of induction arguments, as we shall see throughout this section.

Theorem 2.1. *The arc $v_{2^{m+1}}v_0$ is not on any hamiltonian path of the tournament $T_{[m]}$.*

Proof. The proof is by induction. For $m = 1$, $T_{[1]}$ is an upset tournament on 5 vertices, and it is easy to verify that v_5v_0 is not on any hamiltonian path of $T_{[1]}$. Now, assume the result for m and consider $T_{[m+1]}$. Consider P, a path in $T_{[m+1]}$ containing the arc $v_{2^{m+2}}v_0$ of maximal length, and without loss of generality, assume that $v_{2^{m+1}}$ precedes $v_{2^{m+2}}$ on P. Let v_t be the last vertex of P and v_i the first vertex of P with index $i \leq 2^{m+1}$. Since $v_{2^{m+1}}$ separates v_a from v_b for each $a < 2^m < b$, every vertex between v_i and $v_{2^{m+1}}$ must have index $j < 2^{m+1}$. Similarly, every vertex between $v_{2^{m+1}}$ and $v_{2^{m+2}}$ must have index $j > 2^{m+1}$ and every vertex following v_0 must have index $j < 2^{m+1}$. We now consider the sequence of vertices $Q = v_i, \ldots, v_{2^{m+1}}, v_0, \ldots, v_t$. This sequence contains every vertex of $T_{[m]}$ with index $j \leq 2^{m+1}$ and is a path of $T_{[m]}$. Since Q contains the arc $v_{2^{m+1}}v_0$, by the induction hypothesis, this can not be a hamiltonian path of $T_{[m]}$, and so there is a vertex v_j that is not on the path Q. Consequently, v_j is not on the path P, and P is not a hamiltonian path. □

Thus for any m, we can construct a tournament with $2^{m+1}+1$ vertices with a non-traceable arc xy such that there exist m arc-disjoint paths from y to x. Next we show that this is the minimal number of vertices among strong tournaments with this property.

Lemma 2.1. *If T is a strong n-tournament containing a non-traceable arc xy such that there exist m arc-disjoint paths from y to x, then $n \geq 2^{m+1} + 1$.*

Proof. Again, the proof is by induction. For $m = 1$, the result is obtained by observing that the unique strong tournaments on 3 and 4 vertices are arc-traceable. Next, assume the result for m and consider the smallest strong n-tournament T, with non-traceable arc xy and $m+1$ arc-disjoint paths from y to x. Assume that $n < 2^{m+2}+1$.

As xy is non-traceable, T must have the structure given by Theorem 1.1. Furthermore, the minimality of T implies that $T-z$ has exactly four strong components, and that the second and third components both consist of a single vertex. Let X (respectively, Y) be the set of vertices in the first and second (respectively, third and fourth) strong components of $T-z$. Clearly, either $|X| < 2^{m+1}$ or $|Y| < 2^{m+1}$. Without loss of generality, assume $|X| < 2^{m+1}$ and consider the tournament $T' = T[X \cup \{z\}]$, reversing the arc zx in this tournament if necessary. There exist $m+1$ arc disjoint paths from z to x in T and at most one of these contains the arc zx (if such an arc exists), so there are at least m arc-disjoint paths from z to x in T'. Clearly T' is strong and has fewer than $2^{m+1}+1$ vertices, and so by the induction hypothesis, xz is on some hamiltonian path P of this tournament. We split P into two smaller paths P_1, consisting of all the vertices up to and including x and P_2, consisting of all the vertices of P that follow z (the structure of T guarantees that both P_1 and P_2 are paths of order at least 1). The only vertex of T' on neither P_1 or P_2 is z, so each vertex of X must be on either P_1 or P_2. Next, let C be a hamiltonian cycle of the tournament $T[Y \cup \{z\}]$ (which must be strong, again due to the structure of T) and let Q_1 be the sub-path of C from y to z and Q_2 the sub-path of C from the vertex immediately succeeding z to the vertex immediately preceding y (Q_2 may have order 0). Thus, every vertex of $Y \cup \{z\}$ is on either Q_1 or Q_2. We now construct $H = P_1Q_1P_2Q_2$. We claim that H is a hamiltonian path of T. First, the terminal vertex of P_1 is x and the initial vertex of Q_1 is y, and $xy \in A(T)$ by assumption. Next, the terminal vertex of Q_1 is z, and the initial vertex of P_2 is the vertex immediately following z on P. Lastly, the terminal vertex of P_2 is a vertex of X, while the initial vertex of Q_2 (if any) is a vertex of Y, and by definition X dominates Y. So, H is indeed a path of T. That H is hamiltonian is immediate; it includes all of X and Y as well as the vertex z. Finally, H includes the arc xy. But xy is non-traceable, contradicting our assumption that $n < 2^{m+2} + 1$. □

The previous results apply to a particular arc xy and as a result we make no claim about the number of arc-disjoint paths between every two vertices. In fact, for each m, $T_{[m]}$ contains a vertex of in-degree 1 ($v_{(2^{m+1}-1)}$) as well as a vertex of out-degree 1 (v_1), so we have yet to produce even a 2-arc-strong tournament that is not arc-traceable. We now seek to construct such an m-arc-strong non-arc-traceable tournament. Doing so requires only a minor variation on the construction of $T_{[m]}$; we simply increase δ^0 to be at least m.

Specifically, let T' be a tournament on $n+4m-4$ vertices obtained from $T_{[m]}$ by removing the vertices v_1 and $v_{n-1)}$ and adding sets $U_1 = \{u_{1,1}, \ldots, u_{1,2m-1}\}$ and $U_{n-1} = \{u_{n-1,1} \ldots, u_{n-1,2m-1}\}$. Orient the edges incident with vertices of $U_1 \cup U_{n-1}$ such that $T[U_1]$ and $T[U_{n-1}]$ induce regular tournaments, $N^-(u_{n-1,i}) \cap V(T_{[m]}) = \{v_n\}$ and $N^+(u_{1,i}) \cap V(T_{[m]}) = \{v_0\}$ for $1 \leq i \leq 2m-1$.

Lemma 2.2. *The arc $v_n v_0$ of T' is non-traceable.*

Proof. Assume the result is false, so there is some hamiltonian path H of T' that contains the arc $v_n v_0$. Since $N^-(u) \setminus U_{n-1} = \{v_n\}$ for any $u \in U_{n-1}$, any path ending at a vertex of U_{n-1} is either a path of $T[U_{n-1}]$, or includes an arc $v_n w$ for some $w \in U_{n-1}$. Since H can not contain any of these arcs, every sub-path of H ending at a vertex of U_{n-1} is a path of $T[U_{n-1}]$ and hence the first $2m-1$ vertices of H are precisely U_{n-1}. Similarly, every sub-path of H beginning at a vertex of U_1 is a path of $T[U_1]$ and hence the last $2m-1$ vertices of H are precisely U_1. Now, the sub-path Q of P between the last vertex of U_{n-1} and the first vertex of U_1 is also a path of the tournament $T_{[m]}$. Furthermore, $v_{n-1}Qv_1$ is then a hamiltonian path of $T_{[m]}$ containing the arc $v_n v_0$, a contradiction. Thus, $v_n v_0$ of $T'_{[m]}$ is non-traceable in T'. □

Lemma 2.3. *T' is m-arc-strong.*

Proof. It suffices to show that v_0 both reaches and is reached by every other vertex of T' using m arc-disjoint walks.

To show that each vertex reaches v_0 in m different ways, first consider any vertex $u \in U_1$ and let $N^+(u) \cap U_1 = \{u_1, \ldots, u_{m-1}\}$. Clearly, $W_i = uu_iv_0$ for $1 \leq i \leq m-1$ and $W_m = uv_0$ are m arc-disjoint walks from u to v_0. Additionally, for any vertex $v \notin U_1$, $W_i = vu_iv_0$ for $1 \leq i \leq m-1$ and $W_m = vuv_0$ are arc-disjoint walks from v to v_0.

To show that v_0 reaches every other vertex by m arc-disjoint walks, we use the set of m upset paths $U1, \ldots U_m$. Clearly, $U_1, \ldots, U_m$ are arc-disjoint paths from v_0 to v_n. To show that v_0 reaches every vertex of $u \in U_{n-1}$ by m arc-disjoint walks, let $N^-(w) \cap U_{n-1} = \{u_1, \ldots w_{u-1}\}$ and let $W_i = U_iu_iu$ for $1 \leq i \leq m-1$ and $W_m = U_mu$. As above, for $v \notin U_{n-1} \cup \{v_n\}$, let $W_i = U_iw_iv$ for $1 \leq i \leq m-1$ and $W_m = U_mwv$. □

Finally, we conclude this section with a proof that T' has the fewest vertices among all non-arc-traceable m-arc-strong tournaments.

Theorem 2.2. *If T is a non-arc-traceable m-arc-strong n-tournament, then $n \geq 2^{m+1} + 4m - 3$.*

Proof. For $m = 1$, the result is immediate by observing that all strong tournaments are 1 arc-strong and that $2^{1+1} - 4(1) - 3 = 5$ is the size of the smallest non-arc-traceable strong tournament. So we can assume that $m \geq 2$. Let xy be a non-traceable arc of T. Define S_x as the initial strong component of $T - x$ and S_y as the terminal strong component of $T - y$. As T is m arc-strong, $\delta^0_T \geq m$ and $\delta^0_{T-v} \geq m - 1$ for any $v \in V(T)$. Thus, $\delta^-_{T-x} \geq m - 1$ and $\delta^+_{T-y} \geq m - 1$ and consequently $|S_x| \geq 2m - 1$ and $|S_y| \geq 2m - 1$.

Next, we claim that $S_x \cap S_y = \emptyset$. Assume otherwise, and choose $z \in S_x \cap S_y$. Since $z \in S_x$, it reaches every vertex of $T - x$, and since y can not be in S_x (if it is, then we can find a hamiltonian path beginning with the arc xy), it reaches every vertex of $(T - x) - y$. From this, it is clear that the only vertex that z may not reach in $T - y$ is x. But z is in the terminal strong component of $T - y$, and so is not able to reach any vertex in the initial strong component of this tournament. As a result, the initial strong component of $T - y$ must be $\{x\}$. This requires that $\delta^-_{T-y} = 0 < m - 1$, a contradiction.

Lastly, observe that no path from y to x can use any vertex of $S_x \cup S_y$. Thus we can form T' by replacing the entire set S_x with a single vertex u_x and replacing the entire set S_y with a single vertex u_y without disturbing any path from y to x. Thus, there remain m arc-disjoint paths from y to x in T'. Furthermore, if we let x dominate u_x and let u_y dominate y, then T' is strong. By a similar argument to the one used in Lemma 2.2, any hamiltonian path of T' containing the arc xy can be extended to a hamiltonian path of T containing this arc. As xy is not on any hamiltonian path of T by assumption, it is therefore not on any hamiltonian path of T'. Finally, by Lemma 2.1, T' has at least $2^{m+1} + 1$ vertices and so T has at least $2^{m+1} + 1 - 2 + 2(2m - 1) = 2^{m+1} + 4m - 3$ vertices. □

3 The maximal number of non-traceable arcs in a strong tournament

Let T be the upset tournament obtained from a transitive n-tournament by reversing the arcs $v_i v_{i+2}$ for each odd $i < n$. This tournament is an *upset tournament*, a tournament which can be obtained from a transitive tournament by reversing the arcs in some path from source to sink. In [2], this tournament was shown to have $\frac{n^2-4n+3}{8}$ non-traceable arcs. Further, it was shown that this tournament was maximal among all upset tournaments with respect to the number of non-traceable arcs. We now extend this result and prove that all strong n-tournaments have at most $\frac{n^2-4n+3}{8}$ non-traceable arcs. Once again, we use the structure of non-arc-traceable tournaments given by Theorem 1.1. It is helpful in this case to view this structure slightly differently. Instead of fixing an arc xy and finding the separating vertex z, we fix z and look for a non-traceable arc xy. We now give necessary and sufficient conditions for the existence of such an arc xy where y is separated from x by a given z.

Lemma 3.1. *Let T be a strong tournament with $z \in V(T)$. Further, let $T_z^{(2)}$ dominate z and let z dominate $T_z^{(k-1)}$ in T. For $x \in T_z^{(1)}$ and $y \in T_z^{(k)}$, xy is part of some hamiltonian path if and only if (i) the vertices of $T_z^{(1)}$ can be partitioned into paths P_1, Q_1 such that P_1 begins at a vertex dominated by z and Q_1 ends at x or (ii) the vertices of $T_z^{(k)}$ can be partitioned into paths P_k, Q_k such that Q_k begins at y and P_k ends at a vertex that dominates z.*

Proof. We first prove the sufficiency of condition (i). To prove sufficiency for condition (ii), note that this condition is equivalent to condition (i) in the reversal of T.

First, assume that condition (i) holds. Let P_i be a hamiltonian path of $T_z^{(i)}$ for $2 \leq i \leq k-1$, and let Q_k be any path in $T_z^{(k)}$ from y to a vertex that dominates z. Finally, let P_k be a hamiltonian path of $T_z^{(k)} - V(P_k)$. Then $H = Q_1 Q_k z P_1 P_2 P_2 \ldots P_k$ is a hamiltonian path of T containing the arc xy.

For the converse, assume that H is a hamiltonian path of T containing the arc xy. First, observe that H contains at most one other arc uv with $u \in T_z^{(1)}$ and $v \notin T_z^{(1)}$, as z must lie between any two such arcs on H. If H does not contain another arc with this property, then the initial vertex of H must be a vertex of $T_z^{(2)}$. In this case, the portion of the path H that lies in $T_z^{(1)}$ is a hamiltonian path of this sub-tournament that begins at a vertex dominated by z and ends at x. Removing any arc of this sub-path yields two paths that satisfy condition (i). So, we may assume that H contains an arc $xy \neq uv$ with $u \in T_z^{(1)}$ and $v \notin T_z^{(1)}$ and in this case, the portion of H that lies in $T_z^{(1)}$ consists of two vertex disjoint paths, P_1 and Q_1 (assume that P_1 precedes Q_1 on H). If xy

precedes uv on H, then P_1 ends at x and the vertex immediately preceding Q_1 on H must be z, and so condition (i) is satisfied. If uv precedes xy on H, then there must be an arc $u'v'$ with $u' \notin T_z^{(k)}$ and $v' \in T_z^{(k)}$ such that $u'v'$ precedes xy on H. In the reversal of T, we find that yx precedes $v'u'$ on the reversal of H, and condition (i) is satisfied in $\overline{T}$. Since this is equivalent to condition (ii) in T, H must satisfy condition (ii). □

We note that a corollary of this result gives a sufficient condition for non-arc-traceability in strong tournaments. The results from Section 2 clearly show that the converse of this result is false.

Corollary 3.1. *Let T be a strong tournament having the structure given by Theorem 1.1. If $|N^+(z) \cap T_z^{(1)}| = 1$ and $|N^-(z) \cap T_z^{(k)}| = 1$, then T is not arc-traceable.*

Lemma 3.2. *Let T be a strong tournament having the structure given by Theorem 1.1. If $X = \{x : T_z^{(1)}$ can not covered by paths P and Q such that P begins at a vertex dominated by z and Q ends at $x\}$, then $|X| \leq \frac{a+1}{2}$ where $a = |T_z^{(1)}|$. Similarly, $|Y| \leq \frac{b+1}{2}$ for the analogous set Y, where $b = |T_z^{(k)}|$.*

Proof. Assume that $X = \{x_0, x_1, \ldots, x_m\}$. If $|N^+(z) \cap X| \geq 1$, then assume that z dominates x_0. Let P_i be the longest path not containing x_i that begins at a vertex dominated by z. As z dominates x_0 or some some vertex $z^+ \notin X$, P_i is a path containing at least one vertex for each $1 \leq i \leq m$. Let $S_i = T_z^{(1)} \backslash (V(P_i)$ for $1 \leq i \leq m$. We claim first that $S_i \backslash \{x_i\}$ dominates $V(P_i)$. Assume otherwise, and let v be the last vertex along P_i such that v dominates w for some $w \in S_i \setminus \{x_i\}$. If v is the terminal vertex of P_i then $P_i w$ is a longer path than P_i beginning at a vertex dominated by z. Otherwise, by the minimality of v, we can replace the arc vv^+ of P_i with the 2-path vwv^+ and again obtain a path longer than P_i that begins at a vertex dominated by z. Since $T_z^{(1)}$ is strong, S_i must be reachable from $V(P_i)$ and thus some vertex v_i of $V(P_i)$ must dominate x_i.

Now let Q_i be the longest path of $T[S_i]$ that ends at x_i and let $U_i = S_i \backslash V(Q_i)$. Note, that by the definition of X, $U_i \neq \emptyset$ for $1 \leq i \leq m$ and $U_i \subset S_i \setminus \{x_i\}$ so U_i dominates $V(P_i)$. By a similar argument used above, we also note that U_i is dominated by $V(Q_i)$. Additionally, observe that the terminal strong component of $T[U_i]$ contains no vertex of X and hence $U_i \setminus X \neq \emptyset$.

Finally, we claim that $U_i \cap U_j = \emptyset$ for all $i \neq j$. Assume otherwise, and choose $i \neq j$ with $u \in U_i \cap U_j$. Without loss of generality assume that x_i dominates x_j. Since $u \in U_i$, u dominates v_i, and x_i and x_j dominate u. Now, since u dominates v_i and Q_j dominates u, $v_i \notin Q_j$ and so $v_i \in P_j \cup U_j$. Next, since x_i dominates x_j by assumption and Q_j dominates U_j, $x_i \notin U_j$. Further, x_i dominates u and U_j dominates P_j so $x_i \notin P_j$. Thus, $x_i \in Q_j$. But now $x_i \in Q_j \setminus \{x_j\}$ and $v_i \in P_j \cup U_j$ with v_i dominating x_i, contradicting the fact that $Q_j \setminus \{x_j\} \subset S_j \setminus \{x_j\}$ dominates both P_j and U_j.

The above arguments show that $U_i \setminus X \neq \emptyset$ for $1 \leq i \leq m$ and $U_i \cap U_j = \emptyset$ for $i \neq j$, which establishes that

$$|\bigcup_{i=1}^{m} U_i| \geq \sum_{i=1}^{m} |U_i \setminus X| \geq m = |X| - 1.$$

Thus, we have $a \geq |X| + m = 2m + 1 = 2|X| - 1$ and so $|X| \leq \frac{a+1}{2}$.

The identical bound for the set Y is obtained using an identical argument in the reversal of T. $\square$

Corollary 3.2. *Let T be a strong tournament having the structure given by Theorem 1.1. The number of non-traceable arcs from $T_z^{(1)}$ to $T_z^{(k)}$ is at most $\frac{(ab+a+b+1)}{4}$ where $a = |T_z^{(1)}|$ and $b = |T_z^{(k)}|$.*

Proof. Let B be the set of arcs that we wish to count. Let X (respectively, Y) be the set of vertices such that the vertices of $T_z^{(1)}$ (respectively, $T_z^{(k)}$) can not be split into paths P and Q such that P begins at a vertex dominated by z (respectively, P ends at a vertex dominating z) and Q ends at a vertex of X

(respectively, Q begins at a vertex of Y). Clearly, by Lemma 3.1, $|B| = |X||Y|$ and by Lemma 3.2, $|X||Y| = \frac{(a+1)(b+1)}{4}$. □

Theorem 3.1. *If T is a strong n-tournament, then T has at most $\frac{n^2-4n+3}{8}$ non-traceable arcs.*

Proof. The proof is by induction. For $n = 3$, the only strong 3-tournament is arc-traceable and so has at most $0 = \frac{3^2-4(3)+3}{8}$ non-traceable arcs.

For $n > 3$, assume that T is non-arc-traceable. Thus T has the structure given by Theorem 1.1. Let $A = T_z^{(1)}$ and $B = T_z^{(k)}$ with $a = |A|$ and $b = |B|$ and choose $u \in T_z^{(2)}$ and $w \in T_z^{(k-1)}$. Let xy be a non-traceable arc of T. Since neither $T - x$ nor $T - y$ are strong, xy is either an arc of $T[A \cup \{u, z\}]$, an arc of $T[B \cup \{w, z\}]$ or is an arc from A to B.

We claim that every arc of $T[A \cup \{u, z\}]$ or $T[B \cup \{w, z\}]$ that is non-traceable in T is also non-traceable in this sub-tournament. To see this, assume that some arc a is traceable in $T[A \cup \{u, z\}]$, and let P be a hamiltonian path of this tournament containing a. If P ends at a vertex other than z, then we can take Q to be any hamiltonian path of the remaining vertices and then PQ is a hamiltonian path of T containing a. On the other hand, if H ends at z it must end with the arc uz. If a is not the arc uz, then we can take Q to be any hamiltonian path of the remaining vertices that ends at a vertex dominating z and form $(P-z)Qz$, a hamiltonian path of T containing a. Finally, if $a = uz$, then we can let P' be any hamiltonian path of $T_z^{(1)}$ beginning at a vertex dominated by z and Q any path of the remaining vertices of T and form the hamiltonian path aPQ of T. An identical argument in the reversal of T establishes the corresponding result for $T[B \cup \{w, z\}]$.

As both of the sub-tournaments $T[A \cup \{u, z\}]$ and $T[B \cup \{w, z\}]$ are strong, we can apply the induction hypothesis, and hence $T[A \cup \{u, z\}]$ and $T[B \cup \{w, z\}]$ have at most $\frac{(a+2)^2-4(a+2)+3}{8}$ and $\frac{(b+2)^2-4(b+2)+3}{8}$ non-traceable arcs, respectively. Summing these two values, we obtain $\frac{a^2+b^2-2}{8}$.

Finally, by Corollary 3.2, there are at most $\frac{(a+1)(b+1)}{4}$ non-traceable arcs from A to B. Combining and observing that $a+b \leq n-3$, the number of non-traceable arcs in T is at most

$$\begin{aligned}
\frac{a^2+b^2-2}{8}+\frac{ab+a+b+1}{4} &= \frac{(a^2+2ab+b^2)+2(a+b)}{8} \\
&= \frac{(a+b)^2+2(a+b)}{8} \\
&\leq \frac{(n-3)^2+2(n-3)}{8} \\
&\leq \frac{n^2-6n+9+2n-6}{8} \\
&\leq \frac{n^2-4n+3}{8}
\end{aligned}$$

□

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