Behavior Strategies in Extensive Form Games

A behavior strategy for player i in an extensive form game is a collection of probability distributions over the possible actions $A(h)$ at each information set h that is "owned" by player i.

Let H_i be the collection (i.e. the set of sets) of information sets that belong to player i, let $A(h)$ be the set of actions available at the information set h, and let $\Delta A(h)$ be the set of all possible probability distributions over $A(h)$. Then a behavior strategy b_i for player i can be represented by any element of the cartesian product $\times_{h \in H_i} \Delta A(h)$.

With this approach, we can can treat player zero (i.e., nature) the same as every other player, and we can represent pure strategies by choosing probability distributions for each $A(h)$ that assign probability 1 to a single action in $A(h)$ and probability 0 to all other elements of $A(h)$.

When all the players in the game have chosen a behavior strategy profile, we can find the expected outcome of the game as follows. Recall that every edge e of the game tree is an action that is associated to a vertex in some information set h that belongs to one of the players (this is the "upper" vertex when we explicitly draw the game tree). As a result, we can label every edge e with a real number $0 \leq p_e \leq 1$ that represents the probability assigned to the action associated with edge e.

We now use the fact that for each terminal vertex z of the tree, there is a unique path P_z from the root to z. We will think of this path as a sequence of edges, and thus the probability that the game ends at outcome z is

$$q_z = \prod_{e \in P_z} p_e.$$

(Note that we really only need to think of the path P_z as a *set* of edges, because the commutativity of multiplication.)

Clearly, $0 \leq q_z \leq 1$ for each $z \in Z$ (the set of terminal vertices, or equivalently the set of possible outcomes). We now show that in a game with perfect information $\sum_{z \in Z} q_z = 1$. This is also true in games with imperfect information, but the proof is quite a bit more complicated.

We will use induction on the length of the longest path in our game tree (this is called the *height* of the tree), which we will represent with the variable t. When $t = 1$, then P_z is always a single edge which we can call e_z and $q_z = p_{e_z}$. This clearly requires that our tree has a single decision vertex (the root), and so the game involves a single player and this player has only one information set, and thus her behavior strategy consists of one probability distribution. Thus,

$$\sum_{z \in Z} q_z = \sum_{z \in Z} p_{e_z} = 1.$$

For the inductive step, let D be the set of vertices that are at distance one from the root vertex v in our tree and for each $d \in D$ let e_d be the weight assigned to the edge between d and the root vertex v. Next, we consider the sub-tree rooted at d for each vertex $d \in D$, and we define Z_d as the subset of Z that includes the terminal vertices of our tree that are also terminal in the subtree rooted at d. The sets Z_d for every possible $d \in D$ partition the set Z. Using this observation, the proof is complete by concluding that

$$
\begin{aligned}
\sum_{z\in Z} q_z &= \sum_{d\in D}\Big(\sum_{z\in Z_d} q_z\Big)\\
&= \sum_{d\in D}\Big(\sum_{z\in Z_d}\prod_{e\in P_z} p_e\Big)\\
&= \sum_{d\in D}\Big(\sum_{z\in Z_d} p_{e_d}\times \prod_{e\in(P_z\setminus e_d)} p_e\Big)\\
&= \sum_{d\in D}\left(p_{e_d}\times\Big(\sum_{z\in Z_d}\big(\prod_{e\in(P_z\setminus e_d)} p_e\big)\Big)\right)\\
&= \sum_{d\in D}(p_{e_d}\times 1) \text{ (by the induction hypothesis)}\\
&= 1
\end{aligned}
$$

This allows us to determine the expected utility of a game where each player has chosen a behavior strategy as defined above. Let b_i be the behavior strategy of player i and let $\overline{v_z}$ be the $1\times n$ vector of payoffs (in utility) the players resulting from outcome z. Then with q_z defined above, the expected value of the game is

$$E(b_0, b_1, \dots b_n) = \sum_{z\in Z} q_z \times \overline{v_z}.$$

We'll conclude with an example. Consider the simplified game of poker that is described on page 36 of the Davis text (but ignore the game tree Davis gives). We can model the game tree two ways, either representing nature with a single vertex where both cards are dealt (figure one) or modeling the game as if player one receives her card and makes the decision to bet or pass before player two is dealt his card (figure two). We will calculate the expected outcome of the game when player one bluffs (i.e., bets when she has a low card) with probability $\frac{1}{4}$, and when player two calls when he has a low card with probability $\frac{1}{8}$ (we assume that the decks of cards are fair, which completely determines nature's "strategy" and we also assume that each player always bets or calls when they hold a high card).

Using the first tree, we calculate the probability of the outcome represented by the right-most terminal vertex. The path from the root to this vertex is "LL" followed by "P-l" and these edges are assigned weights $\frac{1}{4}$ and $\frac{3}{4}$ by the strategies described above. So the outcome $[00]^T$ is given the weight $\frac{3}{16}$. Repeating this calculation for every terminal vertex and summing we obtain:

$$\frac{3}{16}[0\ 0]^T + \frac{1}{128}[0\ 0]^T + \frac{7}{128}[5-5]^T + \frac{3}{16}[-5\ \ 5]^T + \frac{1}{16}[-8\ \ 8]^T + \frac{7}{32}[5-5]^T + \frac{1}{32}[8-8]^T + \frac{1}{4}[0\ \ 0]^T = [\frac{23}{128}\ \ \frac{-23}{128}]^T$$

With the second tree, the only changes we must make are to the probabilities assigned to the edges representing actions by player zero. Each path in the first tree uses exactly one edge that represents a random element of the game, and that edge has a probability weight of $\frac{1}{4}$. In the second, tree, the outcomes all contain exactly two such edges, each with weight $\frac{1}{2}$. Therefore, the expected outcome of any two behavior strategies b_1 and b_2 will always be the same in both trees.

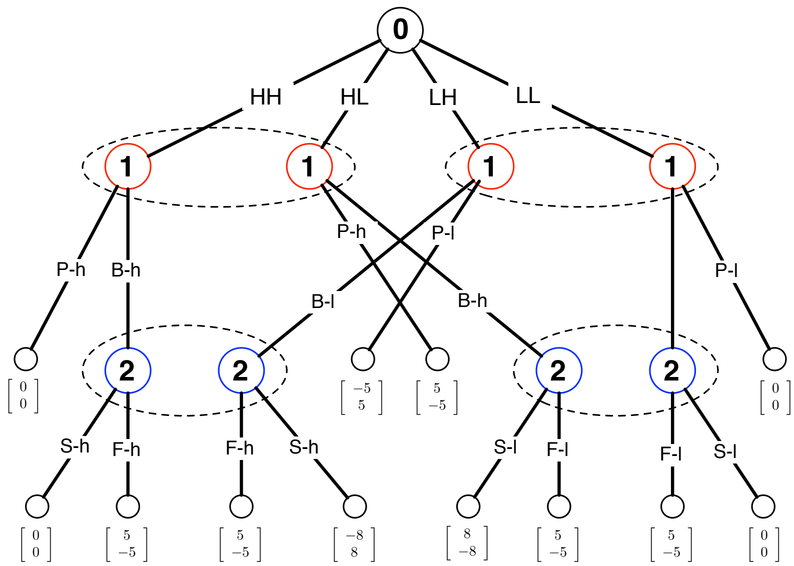

FIGURE 1. An extensive form of simplified poker

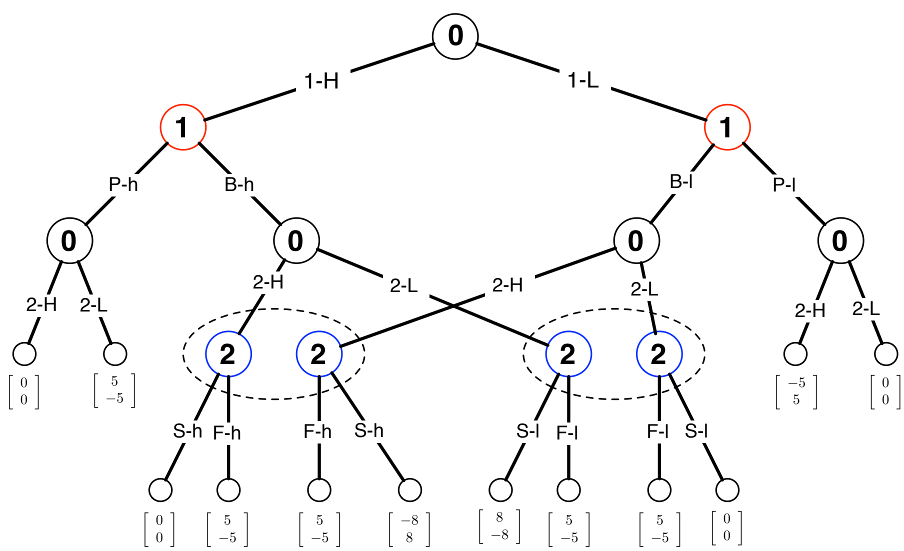

FIGURE 2. An alternate extensive form of simplified poker