A short proof for interval tournaments

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Abstract

We give a short proof of the following result of Brown, Busch and Lundgren: If a tournament T of order n is an interval tournament, then T contains a transitive subtournament of order $n-1$.

For all background and definitions, the reader is referred to [1]. One of the two main results of [1] is the following characterization of interval tournaments.

Theorem 1. *A tournament T or order n is an interval tournament if and only if T contains a transitive subtournament of order $n-1$.*

Sufficiency is easily shown by constructing a zero partition of $A(T)$ when T contains the required transitive subtournament (see Lemma 3.2 in [1]). This supplemental note gives a short proof of the converse, using some basic characteristics of zero partitions.

Acyclic vertex partitions of loopless interval digraphs

Let D be a loopless interval digraph with permuted and zero-partitioned adjacency matrix $A(D)$. For each vertex v let $r(v)$ and $c(v)$ be the row and column of $A(D)$ corresponding to v. Since D is loopless, $a_{r(v)c(v)} = 0$ and this entry is either in C or in R. Let $V_r = \{v \mid a_{r(v),c(v)} \in R\}$ and $V_c = \{v \mid a_{r(v),c(v)} \in C\}$. Note that a given $A(D)$ may have more than one zero partition, and hence V_r and V_c are defined with respect to a particular zero partition.

Observation 2. *Every subset of V_r or of V_c induces an acyclic subgraph of D.*

Proof. Let $S \subseteq V_r$. For $|S| = 1$, the result is immediate. Now let $|S| = m$ and assume the result holds for sets of size $m-1$. Note that the submatrix associated with S is an $m \times m$ matrix which contains an R in every column. Thus, this submatrix contains an R in column one, and the row containing this R must consist of all Rs. This row corresponds to a vertex v with out-degree zero in the digraph induced on S, and so this vertex is not in any cycle of the digraph induced on S. It then follows by induction that $S - \{v\}$ is acyclic, and hence so is S. □

Corollary 3. *The vertex set of every interval tournament can be partitioned into two transitive subtournaments.*

This corollary can be used in many cases to prove that a given tournament is not an interval tournament. As just one example, the Paley tournament (or quadratic residue tournament) of order seven does not contain a transitive subtournament of order four, and thus does not admit a vertex partition into two transitive subtournaments (see [2]).

Next, we show that the unique (up to isomorphism) strong four tournament does not have a zero partition with $|V_r| = |V_c|$.

Observation 4. *No zero partition of the strong four tournament has $|V_c| = 2$.*

Proof. Assume that such a tournament has a zero partition such that $|V_c| = 2$. We use the fact that no 2×2 submatrix of a zero partitionable matrix can contain exactly two R's or exactly two C's unless they share a row or a column.

The (unpartitioned) adjacency matrix of the unique strong four tournament is:

	v_1	v_2	v_3	v_4
v_1	0	1	1	0
v_2	0	0	1	1
v_3	0	0	0	1
v_4	1	0	0	0

If the zero in row one, column four is labeled R, then both the zero in row two and column two and the zero in row three and column three be labeled C; otherwise we have a 2×2 submatrix containing exactly two Rs which are not in the same row or column. So $\{v_2, v_3\} \in V_c$ and since $|V_c| = 2$ by assumption, $V_r = \{v_1, v_4\}$. Thus, we have the following partial labeling of our zero partition.

	v_1	v_2	v_3	v_4
v_1	R	1	1	R
v_2	0	C	1	1
v_3	0	0	C	1
v_4	1	0	0	R

Now observe that the entries shown in blue require that the blue 0 be in C. Similarly, the blue one and the three red entries require that the red 0 be in C as well. But now we have a 2×2 submatrix (rows two and four and columns one and three) with exactly two Cs which appear in distinct rows and columns. This contradiction shows that either $v_1 \in V_c$ or $v_4 \in V_c$ and hence any partition of the adjacency matrix of the strong tournament of order four has $|V_c| = 3$. If the zero in row one, column four is labeled C, then an identical argument with the roles of R and C reversed shows that $|V_r| = 3$. Thus, no zero partition exists with $|V_r| = |V_c| = 2$. □

Characterization of interval tournaments

We can now complete the proof of Theorem 1 for all strong tournaments with the following lemma.

Lemma 5. *If T is a strong interval tournament of order n, then T contains a transitive subtournament or order $n-1$.*

Proof. Let T be an interval tournament, with a zero partition of $A(T)$ which determines sets $V_c = \{y_0, y_1, \dots y_a\}$ and $V_r = \{x_0, x_1, \dots, x_b\}$ indexed in the transitive order (so $x_j \to x_i$ and $y_j \to y_i$ if and only if $j > i$).

First, assume without loss of generality that $x_0 \to y_0$. If $a > 1$ and $b > 1$, then there is a path from y_0 to a vertex in $V_r - \{y_0\}$ since T is strong. Choose a shortest such path P. Note that all internal vertices of P are in V_c, and since this set forms a transitive subtournament, there at most two such internal vertices and P has length two or three. If P has length three, then the four vertices of this path form a strong subtournament consisting of two vertices from each of V_c and V_r. On the other hand if the path has length two (say, $y_0x_jy_i$), then either $y_i \to x_0$ and $y_0x_jy_ix_0y_0$ is a cycle of length four, or $x_0 \to y_i$ and $y_0x_jx_0y_iy_0$ is a cycle of length four. In either case, these four vertices , two each from V_c and V_r, induce a strong subtournament. Having arrived at a contradiction, we conclude that either $a \leq 1$ or $b \leq 1$, and hence T has a transitive subtournament of order $n-1$. □

To complete the characterization of interval tournaments, we now must show the above result holds for tournaments that are not strongly connected. For tournaments T with a single non-trivial strong component of order $k < n$, this is routine. If T is an interval tournament, then so is the non-trivial strong component, and by the theorem above, this subtournament has a transitive subtournament of order $k-1$. This subtournament, combined with the trivial strong components of T, gives a transitive subtournament of order $(k-1)+(n-k) = n-1$. Finally, as noted in [1] (just before Theorem 1.3), no tournament with two or more non-trivial strong components is an interval tournament, since every such tournament contains a subtournament which corresponds to the bipartite graph known as "the insect" which is not an interval bigraph (see [3]).

References

[1] Brown, David E. and Busch, Arthur H. and Lundgren, J. Richard, *Interval Tournaments*, J. Graph Thoery 56 (2007), no. 1, 72–81.

[2] Moon, J., *Topics on Tournaments*, Holt, Rinehart & Winston, New York, 1968.

[3] Müller, H., *Recognizing interval digraphs and interval bigraphs in polynomial time*, Discrete Appl. Math. 78 (1997), no. 1-3, 189–205.