

## Rearrangement on Conditionally Convergent Integrals in Analogy to Series

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### Abstract

Rearrangements on conditionally convergent series suggests the existence of a similar process for integrals, here also referred to as rearrangement. In this document, a general theorem concerning rearrangement for conditionally convergent integrals is presented, as well as supporting theorems and a corollary to the general theorem. The corollary reads: Let  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a continuous function with an everywhere negative and monotone increasing derivative. If  $\int_1^\infty (-1)^x f(x) dx$  is conditionally convergent, then  $\forall z \in \mathbb{C}$ , there exists an arrangement on  $\int_1^\infty (-1)^x f(x) dx$  such that  $z = \int_1^\infty (-1)^x f(x) dx$ .

## 1 Preliminary Theorems and Lemmas

**Note:** Recall that  $e^{i\theta} = \cos(\theta) + i \sin(\theta)$  and  $(-1)^n = e^{in\pi}$ .

**Definition 1.** For any function  $f(x)$  integrable on  $[1, \infty)$  such that

$$\int_1^\infty f(x)dx = \int_0^1 \sum_{j=1}^\infty f(j+u)du$$

a rearrangement on  $\int_1^\infty f(x)dx$  is a rearrangement of the terms in the series  $\sum_{j=1}^\infty f(j+u)$  for each  $u$  on  $[0, 1]$ .

**Theorem 1.** Let  $\{a_j\}_{j=1}^\infty$  and  $\{b_j\}_{j=1}^\infty$  be positive, real, decreasing sequences which converge to zero, where  $a_j \leq b_j$  for all  $j$ . If

$$a_j - a_{j+1} \leq b_j - b_{j+1} \quad j \in \mathbb{N}$$

then

$$\left| \sum_{j=k+1}^\infty (-1)^j a_j \right| \leq \left| \sum_{j=k+1}^\infty (-1)^j b_j \right|.$$

*Proof.* From the hypothesis, both sequences are decreasing, and therefore  $a_j - a_{j+1} > 0$  and  $b_j - b_{j+1} > 0$ . For  $\forall n \in \mathbb{N}$

$$\begin{aligned} \sum_{j=0}^{2n-1} (-1)^j a_{j+1} &= a_l - a_{l+1} + a_{l+2} - a_{l+3} + \dots + a_{l+2n-2} - a_{l+2n-1} \\ &\leq b_l - b_{l+1} + b_{l+2} - b_{l+3} + \dots + b_{l+2n-2} - b_{l+2n-1} \\ &= \sum_{j=0}^{2n-1} (-1)^j b_{j+l}. \end{aligned}$$

with both sums being positive. Since  $a_{2n} \leq b_{2n}$ , it follows that

$$0 < \sum_{j=0}^{2n} (-1)^j a_{j+l} \leq \sum_{j=0}^{2n} (-1)^j b_{j+l}$$

and so  $\forall n \in \mathbb{N}$ ,

$$0 < \sum_{j=0}^n (-1)^j a_{j+l} \leq \sum_{j=0}^n (-1)^j b_{j+l}$$

which implies

$$\left| \sum_{j=l}^{n+l} (-1)^j a_j \right| \leq \left| \sum_{j=l}^{n+l} (-1)^j b_j \right|.$$

Since  $a_j, b_j \rightarrow 0$  as  $j \rightarrow \infty$ , it follows that both series are convergent. Knowing this, let  $l = k + 1$  and  $n \rightarrow \infty$ , to yield that

$$\left| \sum_{j=k+1}^{\infty} (-1)^j a_j \right| \leq \left| \sum_{j=k+1}^{\infty} (-1)^j b_j \right|.$$

□

**Theorem 2.** Let  $f$  be a positive, decreasing function on  $\mathbb{R}^+$ , integrable such that  $\int_1^{\infty} (-1)^x f(x) dx$  is conditionally convergent. Then  $\sum_{j=1}^{\infty} (-1)^j f(j+u)$  is conditionally convergent  $\forall u \in [0, 1]$ .

*Proof.* Note that, for all  $u \in [0, 1]$  and for all  $x \in \mathbb{R}^+$ ,

$$\lceil x \rceil + u < x + 2.$$

This implies

$$f(x+2) < f(\lceil x \rceil + u).$$

Let  $N \in \mathbb{N}$  such that  $N > 2$ . Since

$$\int_0^N f(\lceil x \rceil) dx = \sum_{j=1}^N f(j)$$

it follows that

$$\int_2^N f(x) dx < \sum_{j=1}^N f(j+u).$$

Taking the limit that  $N \rightarrow \infty$  yields that

$$\int_2^{\infty} f(x) dx < \sum_{j=1}^{\infty} f(j+u).$$

Since  $f$  is integrable on  $\mathbb{R}^+$ , it follows that  $\int_1^2 f(x) dx$  is finite. Therefore, if the integral  $\int_1^{\infty} f(x) dx \rightarrow \infty$  then  $\int_2^{\infty} f(x) dx \rightarrow \infty$ , and therefore  $\sum_{j=1}^{\infty} f(j+u) \rightarrow \infty$ .

Suppose that  $\int_1^{\infty} (-1)^x f(x) dx \rightarrow L$ . It then follows  $\forall \epsilon > 0, \exists M > 0$  such that

$$\left| \int_1^y (-1)^x f(x) dx - L \right| < \frac{\epsilon}{\pi\sqrt{2}} \quad y \geq M.$$

Thus, for  $M \leq a \leq b$  it follows by the triangle inequality that

$$\left| \int_a^b (-1)^x f(x) dx \right| \leq \left| \int_1^a (-1)^x f(x) dx - L \right| + \left| \int_1^b (-1)^x f(x) dx - L \right| < \frac{\epsilon\sqrt{2}}{\pi}.$$

Since  $f(x)$  is decreasing, positive, and finite  $\forall x \in \mathbb{R}^+$ , it follows that  $f$  must converge to some value, say  $c$ , as  $x \rightarrow \infty$ . Since  $c \leq f(x)$ , it follows for  $x \in [2n, 2n + 1/2]$  with  $n \in \mathbb{N}$  that

$$-\cos(\pi x)f(x) \leq c \cos(\pi x) \leq \cos(\pi x)f(x)$$

from which follows that

$$\left| \int_{2n}^{2n+1/2} c \cos(\pi x) dx \right| \leq \left| \int_{2n}^{2n+1/2} \cos(\pi x) f(x) dx \right|.$$

Knowing that

$$\int_{2n}^{2n+1/2} \cos(\pi x) dx = \frac{1}{\pi}$$

it follows that

$$\frac{c}{\pi} \leq \left| \int_{2n}^{2n+1/2} \cos(\pi x) f(x) dx \right|.$$

A similar argument will show that the same is true for the sine function. From this, it follows that

$$\frac{\sqrt{2}c}{\pi} \leq \left| \int_{2n}^{2n+1/2} (-1)^x f(x) dx \right|.$$

Let  $n \geq M$ . Then it follows that

$$\frac{\sqrt{2}c}{\pi} \leq \left| \int_{2n}^{2n+1/2} (-1)^x f(x) dx \right| < \frac{\epsilon\sqrt{2}}{\pi} \quad n \geq M$$

and therefore  $c < \epsilon$ . Therefore  $c = 0$ . Thus,  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$ , and therefore  $f(u + j) \rightarrow 0, \forall u \in [0, 1]$  as  $j \rightarrow \infty$ . From this, one can conclude that the series  $\sum_{j=1}^{\infty} (-1)^j f(j + u)$  is convergent. Thus, if  $\int_1^{\infty} (-1)^x f(x) dx$  is conditionally convergent, then  $\sum_{j=1}^{\infty} (-1)^j f(j + u)$  is conditionally convergent  $\forall u \in [0, 1]$ .  $\square$

**Lemma 1.** *Let  $f$  be a positive, decreasing, integrable function on every finite interval of  $\mathbb{R}^+$ . Then*

$$\int_0^1 f(u + j) \cos(\pi u) du \geq 0$$

for  $\forall j \in \mathbb{N}$ .

*Proof.* Note that

$$\begin{aligned} \int_0^1 f(u + j) \cos \pi u du &= \int_0^{1/2} f(u + j) \cos \pi u du + \int_{1/2}^1 f(u + j) \cos \pi u du \\ &\geq \int_0^{1/2} f(1/2 + j) \cos \pi u du + \int_{1/2}^1 f(1/2 + j) \cos \pi u du \\ &= f(1/2 + j) \int_0^1 \cos \pi u du \\ &= 0. \end{aligned}$$

□

**Lemma 2.** Let  $f(x)$  be positive, decreasing, and integrable on every finite subinterval of  $\mathbb{R}^+$  such that

- $\forall u \in [0, 1]$  and  $\forall j \in \mathbb{N}$ ,  $f(u + j) - f(u + j + 1) \leq f(j) - f(j + 1)$ ;
- $\forall u \in [0, 1]$  the series  $\sum_{j=0}^{\infty} (-1)^j f(j + u)$  is convergent.

Then  $(-1)^u \sum_{j=k}^{\infty} (-1)^j f(j + u)$  is integrable on  $[0, 1]$  for each positive  $k$ , and

$$\sum_{j=1}^{\infty} \int_0^1 (-1)^{u+j} f(u + j) du = \int_0^1 \sum_{j=1}^{\infty} (-1)^{u+j} f(u + j) du.$$

*Proof.* First to show integrability. Since  $\sum_{j=k}^{\infty} (-1)^j f(j + u)$  is convergent it follows that  $f(j + u) \rightarrow 0$  as  $j \rightarrow \infty$ . Thus, from the proof of Theorem 1  $\forall n, k \in \mathbb{N}$  with  $n \geq k$ ,

$$\left| \sum_{j=k}^n (-1)^j f(u + j) \right| \leq \left| \sum_{j=k}^n (-1)^j f(j) \right|.$$

It follows from the hypothesis that  $f$  is integrable on  $[j, j + 1]$ , and therefore  $f(u + j)$  is integrable for  $u \in [0, 1]$ . Fix  $k$ . Define

$$h_n(u) = \sum_{j=k}^n (-1)^j f(u + j).$$

Since the finite sum of integrable functions is integrable,  $h_n(u)$  is integrable for each  $n$ . Therefore  $\forall u \in [0, 1]$ , there exists a finite  $h(u) \in \mathbb{R}$  such that  $\lim_{n \rightarrow \infty} h_n(u) = h(u)$ . Also  $|h_n(u)| \leq |h_n(0)|$ . Choose

$$M = \sup\{|h_k(0)|, |h_{k+1}(0)|, \dots\}.$$

Since  $h_n(0)$  is convergent and always finite, such a number exists. Therefore,  $|h_n(u)| \leq M$  for  $\forall u \in [0, 1]$  and  $\forall n \geq k$ . Therefore, by Lebesgue Dominated Convergence Theorem,  $h(u)$  is integrable on  $[0, 1]$ .

Note that  $(-1)^u h(u) = \cos(\pi u)h(u) + i \sin(\pi u)h(u)$ , and that  $\cos(\pi u)$  and  $\sin(\pi u)$  are integrable  $[0, 1]$ . Since the product of two integrable functions is integrable, it follows that  $\cos(\pi u)h(u)$  and  $\sin(\pi u)h(u)$  are integrable  $[0, 1]$ , and therefore  $(-1)^u h(u)$  is integrable on  $[0, 1]$ .

Now, in order to conserve space, define  $g(x) = (-1)^x f(x)$ . Thus, for any finite  $k$

$$\sum_{j=1}^{\infty} \int_0^1 g(u + j) du = \sum_{j=1}^k \int_0^1 g(u + j) du + \sum_{j=k+1}^{\infty} \int_0^1 g(u + j) du$$

and

$$\begin{aligned} \int_0^1 \sum_{j=1}^{\infty} g(u+j) du &= \int_0^1 \sum_{j=1}^k g(u+j) du + \int_0^1 \sum_{j=k+1}^{\infty} g(u+j) du \\ &= \sum_{j=1}^k \int_0^1 g(u+j) du + \int_0^1 \sum_{j=k+1}^{\infty} g(u+j) du. \end{aligned}$$

Therefore

$$\left| \sum_{j=k+1}^{\infty} \int_0^1 g(u+j) du - \int_0^1 \sum_{j=k+1}^{\infty} g(u+j) du \right| = \left| \sum_{j=1}^{\infty} \int_0^1 g(u+j) du - \int_0^1 \sum_{j=1}^{\infty} g(u+j) du \right|$$

for any finite positive  $k$ .

Let  $u \in [0, 1]$ . Since  $f$  is decreasing and positive, it follows that  $0 < f(u + j) - f(u + j + 1)$ , and thus,

$$0 < f(u + j) - f(u + j + 1) \leq f(j) - f(j + 1).$$

Therefore, by Theorem 1

$$\left| \sum_{j=k+1}^{\infty} (-1)^j f(u + j) \right| \leq \left| \sum_{j=k+1}^{\infty} g(j) \right|.$$

Thus,

$$- \sum_{j=k+1}^{\infty} g(j) \leq \sum_{j=k+1}^{\infty} (-1)^j f(j + u) \leq \sum_{j=k+1}^{\infty} g(j)$$

from which follows

$$\left| \int_0^1 \cos(\pi u) \sum_{j=k+1}^{\infty} (-1)^j f(u + j) du \right| \leq \left| \int_0^1 \cos(\pi u) \sum_{j=k+1}^{\infty} g(j) du \right|$$

with the integrability of the left-hand side being given by Lemma 2. Evaluating the right-hand integral,

$$\left| \int_0^1 \cos(\pi u) \sum_{j=k+1}^{\infty} (-1)^j f(u + j) du \right| \leq \frac{1}{\pi} \left| \sum_{j=k+1}^{\infty} g(j) du \right|.$$

The same follows identically for the sine function in place of the cosine function.

Therefore, by the Triangle Inequality,

$$\left| \int_0^1 \sum_{j=k+1}^{\infty} g(u + j) du \right| \leq \frac{\sqrt{2}}{\pi} \left| \sum_{j=k+1}^{\infty} g(j) du \right|.$$

Now, since  $|\cos \pi u| \leq 1$ , it follows that

$$-f(j) \leq -f(u+j) \leq f(u+j) \cos(\pi u) \leq f(u+j) \leq f(j).$$

Integrating  $u$  over  $[0, 1]$  yields

$$-f(j) \leq \int_0^1 f(u+j) \cos(\pi u) du \leq f(j).$$

Similarly,  $|\int_0^1 f(u+j) \sin(\pi u) du| \leq f(j)$ . By a similar argument,

$$-[f(j) - f(j+1)] \leq [f(u+j) - f(u+j+1)] \cos(\pi u) \leq f(j) - f(j+1)$$

and therefore

$$-[f(j) - f(j+1)] \leq \int_0^1 [f(u+j) - f(u+j+1)] \cos(\pi u) du \leq f(j) - f(j+1)$$

with similar following for the sine function. Note, for  $u \in [0, 1]$ ,  $0 \leq \sin(\pi u) \leq 1$ , and therefore  $0 \leq [f(u+j) - f(u+j+1)] \sin(\pi u)$ . Thus,

$$0 \leq \int_0^1 [f(u+j) - f(u+j+1)] \sin(\pi u) du.$$

By Lemma 1, for  $\forall j \in \mathbb{N}$ ,

$$0 \leq \int_0^1 f(u+j) \cos(\pi u) du.$$

For every  $u \in [0, 1]$  and  $y \in \omega$

$$[f(u+j) - f(u+j+1)] \cos \pi u \leq f(u+j) - f(u+j+1) \leq f(j) - f(j+1)$$

. Therefore

$$\int_0^1 [f(u+j) - f(u+j+1)] \cos \pi u du \leq \int_0^1 [f(j) - f(j+1)] du = f(j) - f(j+1)$$

or

$$\int_0^1 f(u+j) \cos \pi u du - \int_0^1 f(u+j+1) \cos \pi u du \leq f(j) - f(j+1)$$

. Letting  $a_j = \int_0^1 f(u+j) \cos \pi u du$  and  $b_j = f(j)$ , it follows from Theorem 1 that

$$\left| \sum_{j=k+1}^{\infty} (-1)^j \int_0^1 f(u+j) \cos \pi u du \right| \leq \left| \sum_{j=k+1}^{\infty} (-1)^j f(j) \right|$$

for all  $k$ . Similarly,

$$\left| \sum_{j=k+1}^{\infty} (-1)^j \int_0^1 f(u+j) \sin \pi u du \right| \leq \left| \sum_{j=k+1}^{\infty} (-1)^j f(j) \right|$$

By the Triangle Inequality

$$\left| \sum_{j=k+1}^{\infty} \int_0^1 g(u+j)du \right| \leq \sqrt{2} \left| \sum_{j=k+1}^{\infty} g(j) \right|$$

and, again by the Triangle Inequality,

$$\left| \sum_{j=k+1}^{\infty} \int_0^1 g(u+j)du - \int_0^1 \sum_{j=k+1}^{\infty} g(u+j)du \right| \leq \sqrt{2} \left(1 + \frac{1}{\pi}\right) \left| \sum_{j=k+1}^{\infty} g(j) \right|.$$

Since  $\sum_{j=1}^{\infty} g(j)$  is convergent, it follows that it is Cauchy, and therefore  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  such that

$$\left| \sum_{j=k+1}^{\infty} g(j) \right| < \frac{\epsilon}{\sqrt{2} \left(1 + \frac{1}{\pi}\right)} \quad k \geq N.$$

Thus

$$\left| \sum_{j=k+1}^{\infty} \int_0^1 g(u+j)du - \int_0^1 \sum_{j=k+1}^{\infty} g(u+j)du \right| < \epsilon \quad k \geq N$$

which implies

$$\left| \sum_{j=1}^{\infty} \int_0^1 g(u+j)du - \int_0^1 \sum_{j=1}^{\infty} g(u+j)du \right| < \epsilon$$

and therefore

$$\sum_{j=1}^{\infty} \int_0^1 g(u+j)du = \int_0^1 \sum_{j=1}^{\infty} g(u+j)du.$$

□

## 2 Main Theorem

**Theorem 3** (Main Theorem). *Let  $f$  be a positive, decreasing function integrable on any finite subinterval of  $\mathbb{R}^+$ . If*

- $\forall u \in [0, 1]$  and  $\forall j \in \mathbb{N}$  it holds that  $f(u+j) - f(u+j+1) \leq f(j) - f(j+1)$
- $\int_1^{\infty} (-1)^x f(x) dx$  is conditionally convergent

then  $\forall z \in \mathbb{C}$  there exists a rearrangement on  $\int_1^{\infty} (-1)^x f(x) dx$  such that ,  $z = \int_1^{\infty} (-1)^x f(x) dx$ .

*Proof.* Note

$$\int_1^{\infty} (-1)^x f(x) dx = \sum_{j=1}^{\infty} \int_j^{j+1} (-1)^x f(x) dx.$$

Making the substitution that  $u = j + x$ , it follows that

$$\int_1^{\infty} (-1)^x f(x) dx = \sum_{j=1}^{\infty} \int_0^1 (-1)^{u+j} f(u+j) du.$$

By Theorem 2,  $\forall u \in [0, 1]$  it holds that  $\sum_{j=1}^{\infty} (-1)^j f(j+u)$  is conditionally convergent. Therefore, by Lemma 2

$$\int_1^{\infty} (-1)^x f(x) dx = \int_0^1 \sum_{j=1}^{\infty} (-1)^{u+j} f(u+j) du.$$

Since  $\sum_{j=1}^{\infty} f(u+j)(-1)^j$  is conditionally convergent  $\forall u \in [0, 1]$  it follows that  $\forall h(u) \in \mathbb{R}$ , there exists a rearrangement of the terms in the series such that  $h(u) = \sum_{j=1}^{\infty} f(u+j)(-1)^j$ . This constitutes a rearrangement on  $\int_1^{\infty} (-1)^x f(x) dx$ .

Choose  $z \in \mathbb{C}$  such that  $z = |z|e^{i\theta}$ .

Now choose a rearrangement of the terms in  $\sum_{j=1}^{\infty} f(u+j)(-1)^j$  for  $\forall u \in [0, 1]$  such that

$$\sum_{j=1}^{\infty} f(u+j)(-1)^j = 2|z| \cos(\pi u - \theta).$$

Then

$$\int_1^{\infty} (-1)^x f(x) dx = z.$$

□

**Corollary 1.** *Let  $f$  be a positive function, integrable on any finite subinterval of  $\mathbb{R}^+$ , with an everywhere negative and increasing derivative. If  $\int_1^{\infty} (-1)^x f(x) dx$  is conditionally convergent, then  $\forall z \in \mathbb{C}$ , there exists a rearrangement on  $\int_1^{\infty} f(x) dx$  such that  $z = \int_1^{\infty} (-1)^x f(x) dx$ .*

*Proof.* It suffices to show that  $f$  satisfies the requirements of the Main Theorem. The requirement of conditional convergence is obviously met. It is also clear that  $f$  is decreasing.

Since  $f$  is continuous, it follows by the Mean Value Theorem that there exists  $c_j \in (j, j+u)$  such that

$$f(u+j) - f(j) = f'(c_j)u.$$

Since  $(j, j+u) \cap (j+1, j+u+1) = \emptyset$ , it follows that  $c_j < c_{j+1}$ , and  $f'(c_j) < f'(c_{j+1})$ . Thus,

$$f(u+j) - f(j) < f(u+j+1) - f(j+1)$$

and

$$f(u + j) - f(u + j + 1) < f(j) - f(j + 1).$$

Thus,  $f$  satisfies the requirements of the Main Theorem.

□