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GLOBAL WELL-POSEDNESS AND ASYMPTOTIC BEHAVIOR OF A CLASS OF INITIAL-BOUNDARY-VALUE PROBLEM OF THE KORTEWEG-DE VRIES EQUATION ON A FINITE DOMAIN

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Abstract. In this paper, we study a class of initial boundary value problem (IBVP) of the Korteweg-de Vries equation posed on a finite interval with nonhomogeneous boundary conditions. The IBVP is known to be locally well-posed, but its global L^2- *a priori* estimate is not available and therefore it is not clear whether its solutions exist globally or blow up in finite time. It is shown in this paper that the solutions exist globally as long as their initial value and the associated boundary data are small, and moreover, those solutions decay exponentially if their boundary data decay exponentially

1. **Introduction.** Considered herein is an initial-boundary value problem (IBVP) for the Korteweg-de Vries equation posed on a finite interval $(0, L)$ with nonhomogeneous boundary conditions, namely,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \qquad x \in (0, L),\ t > 0, \\ u(x, 0) = \phi(x), \\ u(0, t) = h_1(t),\ u_x(L, t) = h_2(t),\ u_{xx}(L, t) = h_3(t). \end{cases} \tag{1.1}$$

This IBVP was considered by Colin and Ghidaglia in 2001 [9] as a model for propagation of surface water waves in the situation where a wave-maker is putting energy in a finite-length channel from the left $(x = 0)$ while the right end $(x = L)$

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of the channel is free (corresponding the case of $h_2 = h_3 = 0$). In particular, they studied the IBVP (1.1) for its well-posedness in the space $H^s(0,L)$ and obtained the following results.

Theorem A:

(i) *Given $h_j \in C^1([0,\infty))$, $j = 1,2,3$ and $\phi \in H^1(0,L)$ satisfying $h_1(0) = \phi(0)$, there exists a $T > 0$ such that the IBVP (1.1) admits a solution (in the sense of distribution)*

$$u \in L^\infty(0,T;H^1(0,L)) \cap C([0,T];L^2(0,L)).$$

(ii) *Assuming $h_1 = h_2 = h_3 \equiv 0$, then for any $\phi \in L^2(0,L)$, there exists a $T > 0$ such that the IBVP (1.1) admits a unique weak solution*

$$u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L)).$$

The result is temporally local in the sense that the solution u is only guaranteed to exist on the time interval $(0,T)$, where T depends on the size of the initial value ϕ and the boundary data $h_j, j = 1,2,3$ in the space $H^1(0,L)$ (or $L^2(0,L)$) and $C_b^1(0,\infty)$, respectively. A problem arises naturally.

Problem B: *Does the solution exist globally?*

Usually, with the local well-posedness in hand, one needs to establish certain global *a priori estimate* of the solutions to obtain the global well-posedness. However, this task turns out to be surprisingly difficult and challenging since the L^2-energy of the solution u of the IBVP (1.1) is not conserved as in the situation of the KdV equation posed on the whole line $\mathbb{R}$ or on a periodic domain $\mathbb{T}$ even in the case of homogeneous boundary conditions ($h_j \equiv 0,\ j = 1,2,3$). Indeed, for any smooth solution u of the IBVP (1.1) with $h_j \equiv 0,\ j = 1,2,3$, it holds that

$$\frac{d}{dt}\int_0^L u^2(x,t)dx = -u^2(L,t) - u_x^2(0,t) - \frac{2}{3}u^3(L,t).$$

The lack of an effective means to deal with the term $\frac{2}{3}u^3(L,t)$ makes it hard to establish the needed global *a priori* estimate for the solutions of the IBVP (1.1) in the space $L^2(0,L)$. Consequently, Problem B is open even for the homogeneous IBVP (1.1).

In [9], Colin and Ghidaglia provided a partial answer to Problem B by showing that *the solution u of the IBVP (1.1) exists globally in $H^1(0,L)$ if the size of its initial value $\phi \in H^1(0,L)$ and its boundary values $h_j \in C^1([0,\infty))$, $j = 1,2,3$ are all small.*

Recently, the IBVP (1.1) has been studied by Kramer and Zhang [24], and Kramer, Rivas and Zhang [26] to address an open question of Colin and Ghidaglia [9] regarding some well-posedness issues of the IBVP (1.1). They obtained the following well-posedness results for the IBVP (1.1) [24, 26].

Theorem C: *Let $s > -\frac{3}{4}$ and $T > 0$ and $r > 0$ be given with*

$$s \neq \frac{2j-1}{2}, \quad j = 1,2,3,\cdots.$$

There exists a $T^*>0$ such that for given $s-$compatible [1]

$$\phi\in H^s(0,L),\quad h_1\in H^{\frac{s+1}{3}}(0,T),\quad h_2\in H^{\frac{s}{3}}(0,T),\quad h_3\in H^{\frac{s-1}{3}}(0,T)$$

satisfying

$$\|\phi\|_{H^s(0,L)}+\|h_1\|_{H^{\frac{s+1}{3}}(0,T)}+\|h_2\|_{H^{\frac{s}{3}}(0,T)}+\|h_3\|_{H^{\frac{s-1}{3}}(0,T)}\le r,$$

the IBVP (1.1) admits a unique solution

$$u\in C([0,T^*];H^s(0,L))\cap L^2(0,T^*;H^{s+1}(0,L)).$$

Moreover, the solution u depends Lipschitz continuously on ϕ and $h_j, j=1,2,3$ in the corresponding spaces.

Remarks:

(1) The well-posedness presented in Theorem C is in its full strength; it includes uniqueness, existence and (Lipschitz) continuous dependence as well as persistence (the solution u forms a continuous flow in the space $H^s(0,L)$).
(2) For the well-posedness of the IBVP (1.1) in the space $H^s(0,L)$, the regularity conditions imposed on the boundary data $h_j,\ j=1,2,3$ are optimal. In particular, when $s=1$, it is only required that $h_1\in H^{\frac{2}{3}}(0,T)$, $h_2\in H^{\frac{1}{3}}(0,T)$ and $h_3\in L^2(0,T)$ instead of $h_1,h_2,h_3\in C^1_b(0,T)$ as in Theorem A.

Nevertheless, the well-posedness result presented in Theorem C is still temporal local. The question whether the solution exists globally remains open. In this paper, we continue to study the IBVP (1.1) but emphasizing on the issues of its global well-posedness in the space $H^s(0,L)$ and the long time asymptotic behavior of those globally existed solutions. In order to describe our results more precisely, we first introduce some notations.

For given $s\ge 0$, $t\ge 0$ and $T>0$, let $\vec{h}:=(h_1,h_2,h_3)$,

$$B^s_{(t,t+T)}=H^{\frac{s+1}{3}}(t,t+T)\times H^{\frac{s}{3}}(t,t+T)\times H^{\frac{s-1}{3}}(t,t+T)$$

and

$$Y^s_{(t,t+T)}=C([t,t+T];H^s(0,L))\cap L^2(t,t+T;H^{s+1}(0,L)),$$
$$X^s_{(t,t+T)}=H^s(0,L)\times B^s_{(t,t+T)}.$$

In addition, let

$$B^s_T=\{\vec{h}\in B^s_{(t,t+T)}\ for\ any\ t\ge 0:\ \sup_{t\ge 0}\|\vec{h}\|_{B^s_{(t,t+T)}}<\infty\},\ X^s_T=H^s(0,L)\times B^s_T,$$

and

$$Y^s_T=\{u\in Y^s_{(t,t+T)}\ for\ any\ t\ge 0:\ \sup_{t\ge 0}\|u\|_{Y^s_{(t,t+T)}}<\infty\}.$$

Both B^s_T and Y^s_T are Banach spaces equipped with the norms

$$\|\vec{h}\|_{B^s_T}:=\sup_{0<t<\infty}\|\vec{h}\|_{B^s_{(t,t+T)}}$$

and

$$\|u\|_{Y^s_T}:=\sup_{0<t<\infty}\|u\|_{Y^s_{(t,t+T)}},$$

[1] The reader is referred to [24] for the precise definition of $s-$compatibility for the IBVP (1.1). One of the sufficient conditions for ϕ,h_1,h_2,h_3 to be $s-$compatible is $\phi\in H^s_0(0,L)$ and

$$h_1\in H_0^{\frac{s+1}{3}}(0,T],\ h_2\in H_0^{\frac{s}{3}}(0,T],\ h_3\in H_0^{\frac{s-1}{3}}(0,T].$$

respectively. If $s=0$, the superscript s will be omitted altogether, so that

$$B_{(t,t+T)}=B^0_{(t,t+T)},\ X_{(t,t+T)}=X^0_{(t,t+T)},\ Y_{(t,t+T)}=Y^0_{(t,t+T)}$$

and

$$B_T=B^0_T,\ X_T=X^0_T,\ Y_T=Y^0_T.$$

Moreover, because of their frequent occurrence, it is convenient to abbreviate the norms of u and h in the space $H^s(0,L)$ and $H^s(a,b)$, respectively, as

$$\|u\|_s=\|u\|_{H^s(0,L)},\qquad |h|_{s,(a,b)}=\|h\|_{H^s(a,b)},$$

and

$$\|u\|=\|u\|_{L^2(0,L)},\qquad |h|_{(a,b)}=\|h\|_{L^2(a,b)}.$$

The main results of this paper are summarized in the following two theorems. The first one states that the small amplitude solutions exist globally.

Theorem 1.1 (Global well-posedness). *Let $s\geq 0$ with*

$$s\neq\frac{2j-1}{2},\quad j=1,2,3,\cdots.$$

There exist positive constants δ and T such that for any $s-$compatible $(\phi,\vec{h})\in X^s_T$ with

$$\|(\phi,\vec{h})\|_{X^s_T}\leq\delta,$$

the IBVP (1.1) admits a unique solution $u\in Y^s_T$.

The second one states that the small amplitude solutions decay exponentially as long as their boundary data decay exponentially.

Theorem 1.2 (Asymptotic behavior). *If, in addition to the assumptions of Theorem 1.1, there exist $\gamma_1>0$ and $C_1>0$ such that*

$$\|\vec{h}\|_{B^s_{(t,t+T)}}\leq C_1e^{-\gamma_1 t}\quad for\ t\geq 0,$$

then there exists γ with $0<\gamma\leq\gamma_1$ and $C_2>0$ such that the corresponding solution u of the IBVP(1.1) satisfies

$$\|u\|_{Y^s_{(t,t+T)}}\leq C_2\|(\phi,\vec{h})\|_{X^s_T}e^{-\gamma t}\quad for\ t\geq 0.$$

The study of the initial-boundary-value problems of the KdV equation posed on the finite domain started as early as in 1979 by Bubnov [6] and has been intensively studied in the past twenty years for its well-posedness following the advances of the study of the pure initial value problems of the KdV equation posed either on the whole line $\mathbb{R}$ or on a torus $\mathbb{T}$. The interested readers are referred to [6, 7, 3, 5, 9, 15, 17, 20, 24, 26] and the references therein for an overall review for the well-posedness of the IBVP of the KdV equation posed a finite domain and [1, 4, 8, 12, 13, 14, 16, 20] for the IBVP of the KdV equation posed on the half line $\mathbb{R}^+$.

The paper is organized as follows.

—- In section 2, we consider the associated linear problem

$$\begin{cases}u_t+(au)_x+u_{xxx}=0, & x\in(0,L),\ t>0,\\ u(x,0)=\phi(x), \\ u(0,t)=h_1(t),\ u_x(L,t)=h_2(t),\ u_{xx}(L,t)=h_3(t).\end{cases}\tag{1.2}$$

where $a = a(x,t)$ is a given function. Attention will be first turned to the situation that $a \equiv 1$ and all boundary data $h_1,\ h_2$ and h_3 are zero:

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & x \in (0,L),\ t > 0, \\ u(x,0) = \phi(x), \\ u(0,t) = 0,\ u_x(L,t) = 0,\ u_{xx}(L,t) = 0. \end{cases} \tag{1.3}$$

It will be shown that the linear KdV equation in (1.3) behaves like a heat equation;

(i) it possesses a remarkably strong smoothing property: for any $\phi \in L^2(0,L)$, the corresponding solution $u(t)$ belongs to the space $H^\infty(0,L)$ for any $t > 0$.
(ii) its solution u decays exponentially in the space $H^s(0,L)$ (for any given $s \geq 0$) as $t \to \infty$.

These heat equation like properties of the IBVP (1.3) enable us to show that for any $s \geq 0$ there exists a $T > 0$ such that if $a \in X_T^s$ and $\|a\|_{X_T^s}$ is small enough, then for any $s-$compatible $(\phi, \vec{h}) \in X_T^s$ with $G(t) = \|\vec{h}\|_{B^s_{(t,t+T)}}$ decays exponentially, the corresponding solution u of (1.2) also decays exponentially in the space $H^s(0,L)$ as $t \to \infty$.

—- In Section 3, the nonlinear IBVP (1.1) will be the focus of our attention. The proofs will be provided for both Theorem 1.1 and Theorem 1.2. As one can see from the proofs, the results presented in Theorem 1.1 and Theorem 1.2 for the nonlinear IBVP (1.1) are more or less small perturbation of the results presented in Section 2 for the linear IBVP (1.2) and therefore are essentially linear results..

—- The paper is ended with concluding remarks given in Section 4. A comparison will be made between the IBVP (1.1) and the following IBVP of the KdV equation posed on $(0,L)$:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & x \in (0,L),\ t > 0, \\ u(x,0) = \phi(x), \\ u(0,t) = 0,\ u(L,t) = 0,\ u_x(L,t) = 0. \end{cases} \tag{1.4}$$

We will see that, although there is just a slight difference between the boundary conditions of the IBVP (1.1) and the IBVP (1.4), there is a big difference between the global well-posedness results for the IBVP (1.1) and the IBVP (1.4). While only small amplitude solutions the IBVP (1.1) exist globally, all solutions of the IBVP (1.4), large or small, exist globally instead.

In addition, the IBVP (1.1) will also be shown in this section, to possess a time periodic solution u^* if the boundary forcing $\vec{h}$ is time periodic with small amplitude. Moreover, this time periodic solution u^* is locally exponentially stable.

2. **Linear problems.** In this section, consideration is first directed to the IBVP of linear KdV equation with homogeneous boundary conditions

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & x \in (0,L),\ t > 0, \\ u(x,0) = \phi(x), \\ u(0,t) = u_x(L,t) = u_{xx}(L,t) = 0. \end{cases} \tag{2.1}$$

Its solution u can be written in the form

$$u(x,t) = W(t)\phi$$

where $W(t)$ is the C^0-semigroup in the space $L^2(0,L)$ generated by the operator

$$Ag := -g''' - g'$$

with domain

$$\boldsymbol{\mathcal{D}}(A) = \{g \in H^3(0,L) : g(0) = g'(L) = g''(L) = 0\}.$$

The following estimate can be found in [24, Proposition 1].

Proposition 2.1. *Let $T > 0$ be given. There exists a constant $C > 0$ depending only on T such that for any $\phi \in L^2(0,L)$,*

$$\|u\|_{Y_{(0,T)}} \leq C_T \|\phi\|.$$

Our main concern is its long time asymptotic behavior. As it holds that, for any smooth solution u of the IBVP (2.1),

$$\frac{d}{dt}\int_0^L u^2(x,t)dx = -u^2(L,t) - u_x^2(0,t),$$

one may wonder if its L^2-energy decays as $t \to \infty$. A detailed spectral analysis of the operator A is needed for the investigation.

Note that both A and its adjoint operator A^* are *dissipative* operator. Indeed, the adjoint operator of A is given by

$$A^* f = f' + f'''$$

with the domain

$$\boldsymbol{\mathcal{D}}(A^*) = \{f \in H^3(0,L) : f(0) = f'(0) = 0, f(L) + f''(L) = 0\}.$$

A direct calculation shows that

$$\int_0^L g(x)(Ag)(x)dx = -\frac{1}{2}((g'(0))^2 + (g(L))^2),$$

$$\int_0^L f(x)(A^*f)(x)dx = -\frac{1}{2}((f(L))^2 + (f'(L))^2)$$

for any $g \in \boldsymbol{\mathcal{D}}(A)$ and $f \in \boldsymbol{\mathcal{D}}(A^*)$. Thus both A and A^* are dissipative operators.

Lemma 2.2. *The spectrum $\sigma(A)$ of A consists of all eigenvalues $\{\lambda_k\}_1^\infty$ with*

$$Re\,\lambda_k < 0, \qquad k = 1, 2, 3, \cdots$$

and

$$\lambda_k = -\frac{8\pi^3 k^3}{3\sqrt{3}L^3} + O(k^2) \qquad as \quad k \to \infty \tag{2.2}$$

Proof. Since A is dissipative and the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$ $(\lambda \in \rho(A))$ is compact,

$$\sigma(A) = \sigma_p(A) = \{\lambda_k\}_1^\infty$$

with $\lambda_k \to \infty$ as $k \to \infty$ and

$$Re\,\lambda_k \leq 0, \quad k = 1, 2, 3, \cdots .$$

We thus need to show that

$$Re\,\lambda_k \neq 0, \quad k = 1, 2, 3, \cdots .$$

Suppose $i\mu \in \sigma_p(A)$ for some real number μ. There exists $0 \neq f = \alpha + i\beta \in \boldsymbol{\mathcal{D}}(A)$ such that

$$\begin{cases} -f'''(x) - f'(x) = i\mu f(x), & x \in (0, L) \\ f(0) = f'(L) = f''(L) = 0 \end{cases} \tag{2.3}$$

or equivalently

$$\begin{cases} -\alpha'''(x) - \alpha'(x) = -\mu\beta(x), & x \in (0, L) \\ \alpha(0) = \alpha'(L) = \alpha''(L) = 0 \end{cases} \tag{2.4}$$

or and

$$\begin{cases} -\beta'''(x) - \beta'(x) = \mu\alpha(x), & x \in (0, L) \\ f(0) = f'(L) = f''(L) = 0 \end{cases} \tag{2.5}$$

Multiply both sides of the equation in (2.4) by α and both sides of the equation in (2.5) by β, and integrate over $(0, L)$. Integration by parts to each resulting equation and add both of them together lead to

$$0 = -\frac{1}{2}|f'(0)|^2 - \frac{1}{2}|f(L)|^2.$$

Consequently,

$$f'(0) = f(L) = 0.$$

Recall that $f(0) = f'(L) = f''(L) = 0$. Therefore, we have $f \equiv 0$. That is a contradiction.

To show that (2.2), we may assume that $Af = -f'''$. The case of $Af = -f''' - f'$ follows from standard perturbation theory (cf. [21]).

Assuming that $Re\,\lambda < 0$. By symmetry, we only need to consider the case that $Im\,\lambda \leq 0$. Denote the three cube roots of $-\lambda$ by $\mu_1,\ \mu_2, \mu_3$. These must have distinct real parts; let μ_1 be the unique root such that $0 \leq arg(\mu_1) \leq \pi/6$ and

$$\mu_2 = e^{\frac{2\pi i}{3}}\mu_1 := \rho\mu_1, \qquad \mu_3 = \rho^2\mu_1.$$

The solution of

$$\begin{cases} \lambda\phi(x) + \phi'''(x) = 0, & x \in (0, L), \\ \\ \phi(L) = \phi'(L) = \phi''(L) = 0 \end{cases} \tag{2.6}$$

is then given by

$$\phi(x) = C_1 e^{\mu_1 x} + C_2 e^{\mu_2 x} + C_3 e^{\mu_3 x}$$

with C_1, C_2 and C_3 satisfying

$$\begin{aligned} C_1 + C_2 + C_3 &= 0, \\ \mu_1 e^{\mu_1 L} C_1 + \mu_2 e^{\mu_2 L} C_2 + \mu_3 e^{\mu_3 L} C_3 &= 0, \\ \mu_1^2 e^{\mu_1 L} C_1 + \mu_2^2 e^{\mu_2 L} C_2 + \mu_3^2 e^{\mu_3 L} C_3 &= 0. \end{aligned}$$

Setting the determinant of the coefficient matrix equal to zero,

$$e^{(\mu_2+\mu_3)L}\mu_2\mu_3(\mu_3 - \mu_2) + e^{(\mu_1+\mu_2)L}\mu_1\mu_2(\mu_2 - \mu_1) + e^{(\mu_3+\mu_1)L}(\mu_1 - \mu_3)\mu_3\mu_1 = 0.$$

By the assumptions, $Re\,\mu_1 > 0$, $Re\,\mu_2 < 0$ and $Re\,\mu_3 \leq 0$. Furthermore,

$$Re\,\mu_1 \to +\infty \quad and \quad Re\,\mu_2 \to -\infty \quad as \quad \lambda \to \infty.$$

Neglecting the term $e^{(\mu_2+\mu_3)L}\mu_2\mu_3(\mu_3-\mu_2)$, which is very small for large λ, we arrive at the equation

$$e^{(\mu_2-\mu_3)L}=\rho+\rho^2,$$

or

$$e^{i\sqrt{3}\mu_1 L}=-1.$$

Therefore,

$$\mu_{1,k}\sim\frac{(1+2k)\pi}{\sqrt{3}L}.$$

As $\lambda_k+\mu^3=0$,

$$\lambda_k=-\mu_{1,k}^3=-\left(\frac{8\pi^3k^3}{3\sqrt{3}L^3}+O(k^2)\right)\quad\text{as } k\to\infty.$$

□

Next lemma gives an asymptotic estimate of the resolvent operator $R(\lambda,A)$ on the pure imaginary axis.

Lemma 2.3.

$$\|R(iw,A)\|=O(|w|^{-2/3}),\quad as\quad |w|\to\infty.$$

Proof. Letting $\lambda\in\rho(A)$ and $f\in L^2(0,L)$ and defining $w=(\lambda I-A)^{-1}f$. In other words, w satisfies

$$\begin{cases}\lambda w+w'''+w'=f,\\ w(0)=w'(L)=w''(L)=0.\end{cases}\tag{2.7}$$

Its solution is given by

$$w(y,\lambda)=\int_0^L G(y,\xi;\lambda)f(\xi)d\xi$$

where $G(y,\xi;\lambda)$, the Green function of (2.7), solves

$$\begin{cases}G'''(y,\xi;\lambda)+G'(y,\xi;\lambda)+\lambda G(y,\xi;\lambda)=\delta(y-\xi)\\ G(0,\xi;\lambda)=G'(L,\xi;\lambda)=G''(L,\xi;\lambda)=0,\end{cases}\tag{2.8}$$

the prime notation representing $\frac{d}{dy}$. With $s_j,\ j=1,2,3$, being the solutions of

$$s^3+s+\lambda=0,$$

$G(y,\xi;\lambda)$ has the form

$$G(y,\xi;\lambda)=\sum_{j=1}^3 c_je^{s_j(y-\xi)}+H(y-\xi)\left(\sum_{j=1}^3\hat{c}_je^{s_j(y-\xi)}\right)$$

with

$$H(x)=\begin{cases}1 & if\ x>0,\\ 0 & if\ x\le 0,\end{cases}$$

the coefficients $\hat{c}_1,\ \hat{c}_2$ and $\hat{c}_3$ satisfying

$$\begin{aligned}\hat{c}_1+\hat{c}_2+\hat{c}_3&=0,\\ \hat{c}_1s_1+\hat{c}_2s_2+\hat{c}_3s_3&=0,\\ \hat{c}_1s_1^2+\hat{c}_2s_2^2+\hat{c}_3s_3^2&=1\end{aligned}$$

and the coefficients $c_1,\ c_2,\ c_3$ solving

$$\begin{pmatrix} 1 & 1 & 1 \\ s_1e^{s_1L} & s_2e^{s_2L} & s_3e^{s_3L} \\ s_1^2e^{s_1L} & s_2^2e^{s_2L} & s_3^2e^{s_3L} \end{pmatrix} \begin{pmatrix} c_1e^{-s_1\xi} \\ c_2e^{-s_2\xi} \\ c_3e^{-s_3\xi} \end{pmatrix} = \begin{pmatrix} 0 \\ d_2 \\ d_3 \end{pmatrix} \tag{2.9}$$

where

$$d_2 = -\left(\hat{c}_1s_1e^{s_1(L-\xi)} + \hat{c}_2s_2e^{s_2(L-\xi)} + \hat{c}_3s_3e^{s_3(L-\xi)}\right), \tag{2.10}$$

$$d_3 = -\left(\hat{c}_1s_1^2e^{s_1(L-\xi)} + \hat{c}_2s_2^2e^{s_2(L-\xi)} + \hat{c}_3s_3^2e^{s_3(L-\xi)}\right). \tag{2.11}$$

A direct computation shows that

$$\hat{c}_1 = \frac{1}{(s_2-s_1)(s_3-s_1)}, \quad \hat{c}_2 = \frac{1}{(s_1-s_2)(s_3-s_2)} \quad \text{and} \quad \hat{c}_3 = \frac{1}{(s_1-s_3)(s_2-s_3)}$$

and

$$c_1 = e^{s_1\xi}\frac{\Delta_1}{\Delta}, \quad c_2 = e^{s_2\xi}\frac{\Delta_2}{\Delta}, \quad c_3 = e^{s_3\xi}\frac{\Delta_3}{\Delta},$$

where Δ is the determinant of the coefficients matrix A of system (2.9),

$$\Delta = e^{s_2+s_3}s_2s_3(s_3-s_2) + e^{s_1+s_3}s_3s_1(s_1-s_3) + e^{s_1+s_2}s_1s_2(s_2-s_1,)$$

and Δ_j be the determinant of the matrix A with the $j-$column replacing by the vector $(0, d_2, d_3)^T$ for $j = 1, 2, 3$. Recall that s_j ($j = 1, 2, 3$) is the solution of

$$s^3 + s + \lambda = 0.$$

If we let $\lambda = i\omega$, then $s_j = i\mu_j$ with μ_j solves

$$\mu^3 - \mu - \omega = 0$$

and therefore,

$$\mu_1 = \alpha + \beta, \quad \mu_2 = \rho^2\alpha + \rho\beta, \quad \mu_3 = \rho\alpha + \rho^2\beta,$$

with $\rho = e^{\frac{2}{3}\pi}$,

$$\alpha = \left(\frac{\omega}{2} + \sqrt{d}\right)^{\frac{1}{3}}, \qquad \beta = \left(\frac{\omega}{2} - \sqrt{d}\right)^{\frac{1}{3}}$$

where

$$d = \frac{\omega^2}{4} - \frac{1}{27}.$$

Thus, as $\omega^{\frac{1}{3}} := b \to \infty$,

$$\begin{aligned} \mu_1 &= b + \frac{1}{3}b^{-1} + O(b^{-5}), \\ \mu_2 &= (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)b - \frac{1}{3}(\frac{1}{2} + \frac{\sqrt{3}}{2}i)b^{-1} + O(b^{-5}), \\ \mu_3 &= (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)b - \frac{1}{3}(\frac{1}{2} - \frac{\sqrt{3}}{2}i)b^{-1} + O(b^{-5}). \end{aligned}$$

As a result, as $b \to \infty$,

$$\begin{aligned} s_1 &= i\mu_1 = ib + O(b^{-1}), \\ s_2 &= i\mu_2 = (-\frac{1}{2}i - \frac{\sqrt{3}}{2})b + O(b^{-1}), \\ s_3 &= i\mu_2 = (-\frac{1}{2}i + \frac{\sqrt{3}}{2})b + O(b^{-1}), \end{aligned}$$

and, asymptotically, as $b \to \infty$,

$$d_2 \sim -s_3 \hat{c}_3 e^{s_3(L-\xi)},$$
$$d_3 \sim -s_3^2 \hat{c}_3 e^{s_3(L-\xi)}.$$

It follows similarly, as $b \to \infty$,

$$\begin{aligned}
\Delta &\sim s_3 s_1 (s_1 - s_3) e^{(s_1+s_3)L},\\
\Delta_1 &\sim -s_3 \hat{c}_3 e^{-s_3(L-\xi)} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ s_3 & 0 & 0 \end{vmatrix}\\
&\sim 0,\\
\Delta_2 &= \begin{vmatrix} 1 & 0 & 1 \\ s_1 e^{s_1 L} & -s_3 \hat{c}_3 e^{s_3(L-\xi)} & s_3 e^{s_3 L} \\ s_1^2 e^{s_1 L} & -s_3^2 \hat{c}_3 e^{s_3(L-\xi)} & s_3^2 e^{s_3 L} \end{vmatrix}\\
&\sim -\hat{c}_3 s_3 e^{s_3(L-\xi)} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ e^{s_1 L} s_1 (s_1 - s_3) & s_3 & 0 \end{vmatrix}\\
&\sim \hat{c}_3 s_3 s_1 (s_1 - s_3) e^{s_3(L-\xi)+s_1 L},\\
\Delta_3 &= \begin{vmatrix} 1 & 1 & 0 \\ s_1 e^{s_1 L} & s_2 e^{s_2 L} & -s_3 \hat{c}_3 e^{s_3(L-\xi)} \\ s_1^2 e^{s_1 L} & s_2^2 e^{s_2 L} & -s_3^2 \hat{c}_3 e^{s_3(L-\xi)} \end{vmatrix}\\
&\sim -\hat{c}_3 s_3 e^{s_3(L-\xi)} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ s_1 e^{s_1 L}(s_1 - s_3) & 0 & s_3 \end{vmatrix}\\
&\sim -\hat{c}_3 s_3 e^{s_3(L-\xi)} s_1 e^{s_1 L} (s_1 - s_3).
\end{aligned}$$

Hence, as $b \to \infty$,

$$c_1 \sim 0, \qquad c_2 \sim \hat{c}_3 e^{(s_2 - s_3)\xi}, \qquad c_3 \sim -\hat{c}_3.$$

Plugging these coefficients in the definition of the Green's function for IBVP (2.8) and considering the case where $y \le \xi$,

$$\begin{aligned}
G(y,\xi; i\omega) &= c_1 e^{s_1(y-\xi)} + c_2 e^{s_2(y-\xi)} + c_3 e^{s_3(y-\xi)}\\
&\sim \hat{c}_3 e^{-s_3\xi + s_2 y} - \hat{c}_3 e^{s_3(y-\xi)}\\
&\sim \hat{c}_3 e^{\frac{i}{2} b(\xi - y) - \frac{\sqrt{3}}{2} b(\xi + y)} - \hat{c}_3 e^{(y-\xi)(\frac{-i}{2} + \frac{\sqrt{3}}{2})b}.
\end{aligned}$$

Therefore, since $\hat{c}_1, \hat{c}_2, \hat{c}_3 \sim b^{-2}$

$$\begin{aligned}
\left| G(y,\xi; ib^3) \right| &\le \hat{c}_3 M_N\\
&\le b^{-2} M_N\\
&\le w^{-2/3} M_N
\end{aligned}$$

for $\{e^{-s_3\xi + s_2 y}, e^{s_3(y-\xi)}\} \le M_N$ for $b > N$. Now, considering the cases where $y > \xi$,

$$
\begin{aligned}
G(y,\xi;i\omega) &= c_1e^{s_1(y-\xi)}+c_2e^{s_2(y-\xi)}+c_3e^{s_3(y-\xi)}+\hat{c}_1e^{s_1(y-\xi)} \\
&\quad +\hat{c}_2e^{s_2(y-\xi)}+\hat{c}_3e^{s_3(y-\xi)} \\
&\sim \hat{c}_3e^{-s_3\xi+s_2y}+\hat{c}_1e^{s_1(y-\xi)}+\hat{c}_2e^{s_2(y-\xi)} \\
&\sim \hat{c}_1e^{s_1(y-\xi)}+\hat{c}_2e^{s_2(y-\xi)}+\hat{c}_3e^{-\frac{\sqrt{3}}{2}b(\xi+y)-i\frac{b}{2}(y-\xi)} \\
&\sim M_Nb^{-2}.
\end{aligned}
$$

Since $\{e^{s_1(y-\xi)},e^{s_2(y-\xi)},e^{-\frac{\sqrt{3}}{2}b(\xi+y)}\}\le M_N$ as $b>N$, so

$$|G(y,\xi,ib^3)|\le M_Nb^{-2}\quad \text{as}\quad b>N,$$

and we can conclude in general that $\forall(y,\xi)$

$$|G(y,\xi,ib^3)|\le M_Nb^{-2}\quad \text{as}\quad b>N.$$

Notice that if we select $\lambda=-i\omega$, the computations are similar and we get the same asymptotic behavior for the Green's function (2.8). □

The following estimate then follows from Lemma 2.2, Lemma 2.3 and [25, Corollary 4.10].

Proposition 2.4. *There exists a $\gamma>0$ such that for any $\phi\in L^2(0,L)$,*

$$u(t)=W(t)\phi\in H^\infty(0,L)\quad \textit{for any } t>0$$

and

$$\|u(t)\|\le Ce^{-\gamma t}\|\phi\|\quad \textit{for all } t\ge 0.$$

Combining Propositions 2.1 and 2.4 gives us

Theorem 2.5. *There exists a $\gamma>0$ such that for $T>0$ there exists a constant $C=C_T>0$ such that*

$$\|u\|_{Y_{(t,t+T)}}\le C_T\|\phi\|e^{-\gamma t}\ \textit{for all } t\ge 0.$$

Now we turn to consider the IBVP of the KdV equation with the nonhomogeneous boundary conditions.

$$
\begin{cases}
u_t+u_x+u_{xxx}=0, & x\in(0,L),\\
u(x,0)=\phi(x),\\
u(0,t)=h_1,\quad u_x(L,t)=h_2,\quad u_{xx}(L,t)=h_3.
\end{cases}
\tag{2.12}
$$

Its solution can be written as

$$u(x,t)=W(t)\phi+W_b(t)\vec{h}$$

where $\vec{h}=(h_1,h_2,h_3)$ and $W_b(t)$ is the boundary integral operator associated to the IBVP (2.12) whose explicit representation formula can be found in [26, Lemma 2.1]. The following estimate is from [26, Prop. 2.2, Prop. 2.3].

Proposition 2.6. *Let $T>0$ be given. There exists a constant $C=C_T$ depending only on T such that for any $\vec{h}\in B_{0,T}$ and $\phi\in L^2(0,L)$, then the IBVP (2.12) admits a unique solution $u\in Y_{(0,T)}$ and, moreover,*

$$\|u\|_{Y_{(0,T)}}\le C_T(\|\phi\|+\|\vec{h}\|_{B_{(0,T)}}).\tag{2.13}$$

Note that the estimate (2.13) can be written as

$$\|u\|_{Y_{(t,t+T)}} \le C_T \|u(\cdot,t)\| + C_T \|\vec{h}\|_{B_{(t,t+T)}} \ \text{for any } t \ge 0$$

because of the semigroup property of the IBVP (2.12).

Attention now is turned to the IVP of a linearized KdV–equation with a variable coefficient $a = a(x,t)$, namely,

$$\begin{cases} u_t + (au)_x + u_x + u_{xxx} = 0, & x \in (0,L), \\ u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \ u_x(L,t) = h_2(t), \ u_{xx}(L,t) = h_3(t). \end{cases} \tag{2.14}$$

The following result is known [24, Prop. 5].

Proposition 2.7. *Let $T > 0$ be given. Assume that $a \in Y_{(0,T)}$. Then for any $\phi \in L^2(0,L), \vec{h} \in B_{(0,T)}$, the IBVP (2.14) admits a unique solution $u \in Y_{(0,T)}$ satisfying*

$$\|u\|_{Y_{(0,T)}} \le \mu(\|a\|_{Y_{(0,T)}}) \left(\|\phi\| + \|\vec{h}\|_{B_{(0,T)}} \right),$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a $T-$dependent continuous nondecreasing function independent of ϕ and $\vec{h}$.

The next theorem presents an asymptotic estimate for solutions of the IBVP (2.14), which will play an important role in studying asymptotic behavior of the nonlinear IBVP in the next section.

Theorem 2.8. *There exists a $T > 0$, $r > 0$ and $\delta > 0$ such that if $a \in Y_T$ with*

$$\|a\|_{Y_T} \le \delta,$$

then for any $\phi \in L^2(0,L), \vec{h} \in B_T$, the corresponding solution u of (2.14) satisfies

$$\|u\|_{Y_{(t,t+T)}} \le C_1 e^{-rt} \|\phi\| + C_2 \|\vec{h}\|_{B_T}$$

for any $t \ge 0$ where C_1 and C_2 are constants independent of ϕ and $\vec{h}$. Furthermore, if

$$\|\vec{h}\|_{B_{(t,t+T)}} \le C_3 e^{-\nu t} \ \text{for all } t \ge 0 \tag{2.15}$$

with constants $\nu > 0$ and $C_3 > 0$, then there exist $0 < \gamma \le \max\{r,\nu\}$ and $C > 0$ such that

$$\|u\|_{Y_{(t,t+T)}} \le C \|(\phi, \vec{h})\|_{X_T} e^{-\gamma t}$$

for any $t \ge 0$.

The following two technical lemmas are needed for the proof of Theorem 2.8.

Lemma 2.9. *Let $T > 0$ be given. There exists a constant $C = C_T > 0$ such that*

(i) for any $u, \ v \in Y_{(0,T)}$,

$$\|(uv)_x\|_{L^1(0,T;L^2(0,L))} \le C_T \|u\|_{Y_{0,T}} \|v\|_{Y_{(0,T)}};$$

(ii) for $f \in L^1(0,T;L^2(0,L))$, let

$$u = \int_0^t W(t-\tau) f(\tau) d\tau,$$

then

$$\|u\|_{Y_{(0,T)}} \le C_T \|f\|_{L^1(0,T;L^2(0,L))}.$$

Lemma 2.10. *Consider a sequence* $\{y_n\}_0^\infty$ *in a Banach space* X *generated by iteration as follows:*

$$y_{n+1} = Ay_n + F(y_n)\,, \quad n = 0, 1, 2, \cdots . \tag{2.16}$$

Here, the linear operator A *is bounded from* X *to* X *with*

$$\|Ay_n\|_X \leq \gamma \|y_n\|_X \tag{2.17}$$

for some finite value γ *and all* $n \geq 0$. *The nonlinear function* F *mapping* X *to* X *is such that there is constant* β *and a sequence* $\{b_n\}_{n\geq 0}$ *for which*

$$\|F(y_n)\|_X \leq \beta \|y_n\|_X^2 + b_n \tag{2.18}$$

for all $n \geq 0$.

(i) If $0 < \gamma < 1$, *then the sequence* $\{y_n\}_{n=0}^\infty$ *defined by (2.16) satisfies*

$$\|y_{n+1}\|_X \leq \gamma^{n+1}\|y_0\|_X + \frac{b^*}{1-\gamma} \tag{2.19}$$

for any $n \geq 1$, *where* $b^* = \sup_{n\geq 0} b_n$.

(ii) If, in addition,

$$b_{n+1} \leq \delta^n c_n$$

with some finite value of δ , *then*

$$\|y_{n+1}\|_X \leq \gamma^{n+1}\|y_0\|_X + nr^n c^* \tag{2.20}$$

for any $n \geq 1$, *where* $r = \max\{\gamma,\ \delta\}$ *and* $c^* = \sup_{n\geq 0} c_n$.

The proof of this Lemma 2.9 can be found in [24, Lemma 3]. As for Lemma 2.10, its proof is similar to that of Lemma 3.2 in [2] with just a minor modification.
Proof of Theorem 2.8: Rewrite (2.14) in its integral form

$$u(t) = W(t)\phi + W_b(t)\vec{h} - \int_0^t W(t-\tau)(au)_x(\tau)d\tau.$$

Thus, for any $T > 0$, using Proposition 2.4 and Proposition 2.6 as well as Proposition 2.7 , Lemma 2.9 and 2.10,

$$\begin{aligned}
\|u(\cdot,T)\| \quad &\leq \quad Ce^{-\beta T}\|\phi\| + C_T\|\vec{h}\|_{B_{(0,T)}} + \int_0^T \|(au)_x\|(\tau)d\tau \\
&\leq \quad Ce^{-\beta T}\|\phi\| + C_T\|\vec{h}\|_{B_{(0,T)}} + C_T\|a\|_{Y_{(0,T)}}\|u\|_{Y_{(0,T)}} \\
&\leq \quad Ce^{-\beta T}\|\phi\| + C_T\|\vec{h}\|_{B_{(0,T)}} \\
&\qquad +C_T\|a\|_{Y_{(0,T)}}\mu(\|a\|_{Y_{(0,T)}})(\|\phi\| + C_T\|\vec{h}\|_{B_{(0,T)}}) \\
&\leq \quad Ce^{-\beta T}\|\phi\| + C_T\|a\|_{Y_{(0,T)}}\mu(\|a\|_{Y_{(0,T)}})\|\phi\| \\
&\qquad +C_T(1 + \|a\|_{Y_{(0,T)}}\mu(\|a\|_{Y_{(0,T)}}))\|\vec{h}\|_{B_{(0,T)}}.
\end{aligned}$$

Note that in the above estimate, the constant C is independent of T. Let

$$y_n = u(\cdot, nT) \ \text{ for } n = 0, 1, 2, \cdots$$

and let v be the solution of the IBVP

$$\begin{cases} v_t + v_x + (av)_x + v_{xxx} = 0, \qquad v(x,0) = y_n(x), \\ \\ v(0,t) = h_1(t+nT),\ v_x(L,t) = h_2(t+nT),\ v_{xx}(L,t) = h_3(t+nT) \end{cases} \tag{2.21}$$

Thus $y_{n+1}(x) = v(x,T)$ by the semigroup property of the system (2.14). Consequently, we have the following estimate for y_{n+1} :

$$\begin{aligned} \|y_{n+1}\| &\leq Ce^{-\beta T}\|y_n\| + C_T\|a\|_{Y_{(nT,(n+1)T)}}\mu(\|a\|_{Y_{(nT,(n+1)T)}})\|y_n\| \\ \\ &+ C_T(1+\|a\|_{Y_{(nT,(n+1)T)}}\mu(\|a\|_{Y_{(nT,(n+1)T)}}))\|\vec{h}|_{B_{(nT,(n+1)T)}} \end{aligned}$$

for $n = 0, 1, 2, \cdots$. Choose T and δ such that

$$Ce^{-\beta T} = \gamma < 1, \quad \gamma + C_T\delta\mu(\delta) := r < 1.$$

Then,

$$\|y_{n+1}\| \leq r\|y_n\| + b_n$$

for all $n \geq 0$ if $\|a\|_{Y_T} \leq \delta$ where

$$b_n = C_T(1+\|a\|_{Y_{(nT,(n+1)T)}}\mu(\|a\|_{Y_{(nT,(n+1)T)}}))\|\vec{h}\|_{B_{(nT,(n+1)T)}} := c_n\|\vec{h}\|_{B_{(nT,(n+1)T)}}.$$

It follows from Lemma 2.10 (with $\beta = 0$) that

$$\|y_{n+1}\| \leq r^n\|y_0\| + \frac{b^*}{1-r}$$

or

$$\|y_{n+1}\| \leq r^n\|y_0\| + n\delta^n c^*$$

with $\delta = e^{-\eta T}$ for any $n \geq 0$ depending if (2.15) holds where $b^* = \sup_{n\geq 0} b_n$ and $c^* = C_3 \sup_{n\geq 0} c_n$. These inequalities imply the conclusion of Theorem 2.8. □

3. **Nonlinear problems.** In this section we consider the IBVP of the nonlinear KdV equation posed on the finite domain $(0,L)$:

$$\begin{cases} u_t + uu_x + u_x + u_{xxx} = 0, \quad x \in (0,L), \\ u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \quad u_x(L,t) = h_2(t), \quad u_{xx}(L,t) = h_3(t). \end{cases} \tag{3.1}$$

According to Theorem B, *for given $(\phi,\vec{h}) \in X_{(0,T)}$, there exists a $T^* \in (0,T]$ such that the IBVP (3.1) admits a unique solution $u \in Y_{(0,T^*)}$.* This well-posedness result is temporally local in the sense that the solution u is only guaranteed to exist on the time interval $(0,T^*)$, where T^* depends on the norm of $(\phi,\vec{h})$ in the space $X_{(0,T)}$. The next proposition presents an alternative view of local well-posedness for the IBVP (3.1). If the norm of $(\phi,\vec{h})$ in $X_{(0,T)}$ is not too large, then the corresponding solution is guaranteed to exist over the entire time interval $(0,T)$.

Proposition 3.1. *Let $T > 0$ be given. There exists $\delta > 0$ such that if $\|(\phi,\vec{h})\|_{X_{(0,T)}} \leq \delta$, then the solution u of the IBVP (3.1) belongs to the space $Y_{(0,T)}$ and, moveover, there exists a constat $C > 0$ depending only on T and δ such that*

$$\|u\|_{Y_{(0,T)}} \leq C\|(\phi,\vec{h})\|_{X_{(0,T)}}.$$

Proof. For $(\phi, \vec{h}) \in X_{(0,T)}$, define a map $\Gamma : Y_{(0,T)} \to Y_{(0,T)}$ by

$$\Gamma(v) = W(t)\phi(x) + W_b(t)\vec{h} - \int_0^t W(t-\tau)\,(vv_x)\,(\tau)d\tau.$$

By Lemma 2.9 and Proposition 2.6, for any v, v_1, $v_2 \in Y_{(0,T)}$,

$$\|\Gamma(v)\|_{Y_{(0,T)}} \le C_1\|(\phi, \vec{h})\|_{X_{(0,T)}} + C_2\|v\|^2_{Y_{(0,T)}}$$

and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Y_{(0,T)}} \le C_2\|(v_1 + v_2\|_{Y_{(0,T)}}\|v_1 - v_2\|_{Y_{(0,T)}}$$

where C_1 and C_2 are constants depending only on T. Choose

$$\delta = \frac{1}{8C_1C_2}, \qquad M = 2C_1\delta.$$

Let

$$S_M = \{v \in Y_{(0,T)},\ \|v\|_{Y_{(0,T)}} \le M\}.$$

Assume that $\|(\phi, \vec{h})\|_{X_{(0,T)}} \le \delta$. Then, for any v, v_1, $v_2 \in M$,

$$\|\Gamma(v)\|_{Y_{(0,T)}} \le C_1\|(\phi, \vec{h})\|_{X_{(0,T)}} + C_2\|v\|^2_{Y_{(0,T)}} \le C_1\delta + 4C_2C_1^2\delta^2 \le \frac{3}{2}C_1\delta \le M$$

and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Y_{(0,T)}} \le 2C_2M\|v_1 - v_2\|_{Y_{(0,T)}} \le \frac{1}{2}\|v_1 - v_2\|_{Y_{(0,T)}}.$$

Thus Γ is a contraction in the ball S_M and its fixed point $u \in Y_{(0,T)}$ is the desired solution of the the IBVP (3.1) which, moreover, satisfies

$$\|u\|_{Y_{(0,T)}} \le \frac{3}{4}C_1\|(\phi, \vec{h})\|_{X_{(0,T)}}.$$

□

Next we show that if the initial value ϕ and boundary value $\vec{h}$ are small, then the corresponding solution u exists for any time $t > 0$ and, moreover, its norm in the space $L^2(0, L)$ is uniformly bounded.

Proposition 3.2. *There exist positive constants T, δ_j, $j = 1, 2$ and r such that if $(\phi, \vec{h}) \in X_T$ satisfying*

$$\|\phi\| \le \delta_1, \quad \|\vec{h}\|_{B_T} \le \delta_2 \tag{3.2}$$

then the corresponding solution u of (3.1) is globally defined and belongs to the space Y_T. Moreover,

$$\|u(\cdot, t)\| \le C_1e^{-rt}\|\phi\| + C_2\|\vec{h}\|_{B_T}$$

for any $t \ge 0$ and

$$\|u\|_{Y_T} \le C_3\|(\phi, \vec{h})\|_{X_T},$$

where C_1, C_2 and C_3 are constants depending only on T, δ_1 and δ_2.

Proof. For given $\phi \in L^2(0, L)$ and $\vec{h} \in B_T$, rewrite the IBVP (3.1) in its integral form

$$u(t) = W(t)\phi + W_b(t)\vec{h} - \int_0^t W(t-\tau)(uu_x)(\tau)d\tau.$$

For any given $T > 0$, there exist $c_1 > 0$ independent of T and c_2, c_3 depending only on T such that for any $0 \le t \le T$,

$$\|u(\cdot, t)\| \le c_1e^{-rt}\|\phi\|_{L^2(0,L)} + c_2\|u\|^2_{Y_{(0,T)}} + c_3\|\vec{h}\|_{B_{0,T}}. \tag{3.3}$$

By Proposition 3.1, there exists a $\delta > 0$ and a constant $c_4 > 0$ such that if

$$\|(\phi, h)\|_{X_{(0,T)}} \leq \delta, \tag{3.4}$$

then

$$\|u\|_{Y_{(0,T)}} \leq c_4 \|(\phi, h)\|_{X_{(0,T)}}.$$

Thus, if (3.4) holds and (3.3) is evaluated at $t = T$,

$$\|u(\cdot, T)\| \leq c_1 e^{-rT}\|\phi\| + c_5(\|\phi\|^2 + \|\vec{h}\|^2_{B_{(0,T)}}) + c_3\|\vec{h}\|_{B_{(0,T)}}$$

with $c_5 = c_2 c_4^2$. Since c_1 is independent of T, one can choose $T > 0$ so that $c_1 e^{-rT} = \gamma < \frac{1}{2}$. Then choose δ_1 and δ_2 such that

$$\delta_1 + \delta_2 \leq \delta$$

and

$$\frac{1}{2}\delta_1 + c_5(\delta_1^2 + \delta_2^2) + c_3\delta_2 \leq \delta_1.$$

For such values of δ_1 and δ_2, we have that

$$\|u(\cdot, T)\| \leq \delta_1,$$

and, in addition, by the assumption,

$$\|\vec{h}\|_{B_{(T,2T)}} \leq \delta_2.$$

Hence repeating the argument, we have that

$$\sup_{T \leq t \leq 2T} \|u(\cdot, t)\| \leq \delta_1, \qquad \|u\|_{Y_{(T,2T)}} \leq c_4\delta.$$

Continuing inductively, it is adduced that

$$\sup_{t \geq 0} \|u(\cdot, t)\| \leq \delta_1, \|u\|_{Y_T} \leq c_4\delta.$$

Let $y_n = u(\cdot, nT)$ for $n = 1, 2, \cdots$. Using the semigroup property of (3.1), one obtains

$$\|y_{n+1}\| \leq \frac{1}{2}\|y_n\| + c_2\|y_n\|^2 + c_3\|h\|_{B_{(nT,(n+1)T)}}$$

for any $n \geq 0$ provide $\|y_0\| \leq \delta_1$ and

$$\sup_{n \geq 0} \|\vec{h}\|_{B_{(nT,(n+1)T)}} \leq \delta_2.$$

By Lemma 2.10, there exists $0 < \nu < 1$, $\delta_1^* > 0$ and $\delta_2^* > 0$ such that if

$$\|y_0\| \leq \delta_1^*, \quad b_n = C_3\|\vec{h}\|_{B_{(nT,(n+1)T)}} \leq \delta_2 \leq \delta_2^*$$

for all $n \geq 0$, then

$$\|y_{n+1}\| \leq \nu^{n+1}\|y_0\| + \frac{b^*}{1-\nu}$$

for all $n \geq 1$, where $b^* = \max_n\{b_n\}$. This leads by standard arguments to the conclusion of Proposition 3.2. □

Proposition 3.3. *Under the assumptions of Proposition 3.2, if*

$$\|\vec{h}\|_{B_{(t,t+T)}} \leq C_4 e^{-\nu t} \text{ for all } t \geq 0$$

with constants $\nu > 0$ and $C_4 > 0$, then there exist a $0 < \gamma \leq \max\{r, \nu\}$ and $C > 0$ such that

$$\|u\|_{Y_{(t,t+T)}} \leq C\|(\phi, \vec{h})\|_{X_T} e^{-\gamma t}$$

for any $t \geq 0$.

Proof. Setting $a = \frac{1}{2}u$, the equation in (3.1) becomes

$$u_t + u_x + (au)_x + u_{xxx} = 0.$$

Proposition 3.3 follows from Proposition 3.2 and Theorem 2.8 as a corollary. □

Now we are at the stage to present the proofs of Theorem 1.1 and Theorem 1.2.
Proof of Theorem 1.1: We only consider the case of $0 \le s \le 3$. The proof for the case of $s > 3$ is similar. Note first the theorem is Proposition 3.2 when $s = 0$. For $s = 3$, let $v = u_t$. Then v solves

$$\begin{cases} v_t + v_x + v_{xxx} + (uv)_x = 0, & x \in (0, L),\ t > 0, \\ v(x, 0) = \phi^*(x), \\ v(0, t) = h_1'(t),\ u_x(L, t) = h_2'(t),\ v_{xx}(L, t) = h_3'(t) \end{cases} \tag{3.5}$$

with $\phi^*(x) = -\phi'(x) - \phi(x)\phi'(x) - \phi'''(x)$. Note that

$$\|u\|_{Y_T} \le C\|(\phi, \vec{h})\|_{X_T}.$$

Thus, invoking Theorem 2.8 yields that $v \in Y_T$ and

$$\|v\|_{Y_T} \le C\|(\phi^*, \vec{h}')\|_{X_T}.$$

Then, it follows from the equation

$$u_{xxx} = -u_t - uu_x - u_x = -v - uu_x - u_x$$

that $u \in Y_T^3$ and

$$\|u\|_{Y_T^3} \le C\|(\phi, \vec{h})\|_{X_T^3}$$

for some constat $C > 0$ Thus, Theorem 1.1 holds for $s = 3$. The case of $0 < s < 3$ then follows by using the nonlinear interpolation theory of Tartar [30, 1]. □
Proof of Theorem 1.2: For any $r > 0$, define the space

$$X_T^s(r) = \{(\phi, \vec{h}) \in X_T^s |\ (\phi, e^{rt}\vec{h}) \in X_T^s\}$$

and

$$Y_T^s(r) = \{u \in Y_T^s |\ e^{rt}u \in Y_T^s\}.$$

Equipped with the norms

$$\|(\phi, \vec{h})\|_{X_T^s(r)} := \|(\phi, e^{rt}\vec{h})\|_{X_T^s}$$

and

$$\|u\|_{Y_T^s(r)} := \|e^{rt}u\|_{Y_T^s},$$

respectively, both $X_T^s(r)$ and $Y_T^s(r)$ are Banach spaces. Theorem 1.2 can then be restated as

For given $s \ge 0$ and $\nu > 0$ with

$$s \ne \frac{2j-1}{2}, \quad j = 1, 2, 3, \cdots,$$

there exist positive constants T, γ, δ and C such that for $s-$compatible $(\phi, \vec{h}) \in X_T^s(\nu)$ with

$$\|(\phi, \vec{h})\|_{X_T^s(\nu)} \le \delta,$$

the corresponding solution u of the IBVP(3.2) belongs to the space $Y_T^s(\gamma)$ and

$$\|u\|_{Y_T^s(\gamma)} \le C\|(\phi, \vec{h})\|_{X_T^s(\nu)}.$$

Its proof is similar to that of Theorem 1.1. □

4. **Concluding remarks.** The focus of our discussion has been the well-posedness of the initial value problem of the KdV equation posed on the finite interval $(0, L)$:

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & x \in (0, L),\ t > 0, \\ u(x, 0) = \phi(x), \\ u(0, t) = h_1(t),\ u_x(L, t) = h_2(t),\ u_{xx}(L, t) = h_3(t). \end{cases} \tag{4.1}$$

It is considered with the initial data $\phi \in H^s(0, L)$ and the boundary data $\vec{h} = (h_1, h_2, h_3)$ belongs to the space $B^s_{(0,T)} = H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s}{3}}(0, T) \times H^{\frac{s-1}{3}}(0, T)$ with $s \geq 0$. Although the IBVP is known to be locally (in time) well-posed, whether solutions exist globally is still an open question because even the simplest global *a priori* L^2- estimate is not available for solutions of the IBVP (4.1). As a partial answer to this open problem of the global well-posedness, we have shown in this article that the solutions exist globally (in time) in the space $H^s(0, L)$ for any $s \geq 0$ as long as its auxiliary data $(\phi, \vec{h})$ is small in the space X^s_T, which is an improvement of Colin and Ghidaglia's early work [9]. In addition, we have studied the long time behavior of those globally existed solutions and have shown that those small amplitude solutions decay exponentially if their boundary value $\vec{h}(t)$ decays exponentially. In particular, those solutions satisfying homogenous boundary conditions decay exponentially in the space $H^s(0, L)$ if their initial values are small in $H^s(0, L)$.

It is interesting to compare the IBVP (4.1) with another well-studied IBVP of the KdV equation posed on the finite interval $(0, L)$:

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & x \in (0, L),\ t > 0, \\ u(x, 0) = \phi(x), \\ u(0, t) = h_1(t),\ u(L, t) = h_2(t),\ u_x(L, t) = h_3(t). \end{cases} \tag{4.2}$$

While the well-posedness results as described by Theorem C, Theorem 1.1 and Theorem 1.2 for the IBVP (4.1) are all true for the IBVP (4.2), the IBVP (4.2) is globally well-posed in the space $H^s(0, L)$ for any $s \geq 0$ [3, 17]:

for given $s-$compatible $(\phi, \vec{h}) \in H^s(0, L) \times H^{\frac{s^*+1}{3}}(0, T) \times H^{\frac{s^*+1}{3}}(0, T) \times H^{\frac{s^*}{3}}(0, T)$, the IBVP (4.2) admits a unique solution $u \in Y^s_{(0,T)}$. Here $s^* = s^+$ if $0 \leq s < 3$ and $s^* = s$ if $s \geq 3$.

The reason for the IBVP (4.2) to be globally well-posedness is simply that global *a priori* L^2 estimate holds for solution u of the IBVP (4.2) with homogeneous boundary conditions;

$$\frac{d}{dt} \int_0^L u^2(x, t) dx + u_x^2(0, t) = 0 \quad for\ any\ t \geq 0.$$

Thus whether solutions of the IBVP (4.1) exist globally or blow up in finite time becomes really interesting. If it does not blow up in finite time, then how to establish its global well-posedness without knowing if its simplest global L^2 *a priori* estimate holds or not? As Archimedes said, "*Give me a place to stand and with a lever I will move the whole world.*" For the IBVP (4.1), if there are no global *a priori* estimates available, how to prove its global well-posedness? On the other hand, if some solutions of the IBVP (4.1) do blow up in finite time, that would be also very interesting since the blow up would be mainly caused by the boundary conditions rather than the nonlinearity of the equation. We are not aware of any such kind of results existed in the literature.

Finally we point out that, started by the work of Ghidaglia [18] in 1988, the KdV equation posed on a finite domain has also been studied intensively from dynamics point of views [2, 18, 19, 29, 34, 33, 35, 36, 37]. One of the questions people are interested is whether the equation admits a time periodic solution if the external forcing functions are time periodic. Such a time periodic solution, if exists, is called *forced oscillation*, which can be viewed as a *limit cycle* from dynamics point of view. A further question to study for this *limit cycle* is: what is its stability? In [33], Usman and Zhang has obtained the following result for the following system associate to the IBVP (4.2):

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & x \in (0, L),\ t > 0, \\ u(0,t) = h_1(t),\ u(L,t) = 0,\ u_x(L,t) = 0. \end{cases} \tag{4.3}$$

Theorem D: *There exists a $\delta > 0$ such that if $h_1 \in H^{\frac{1}{3}}_{loc}(R^+)$ is a time -periodic function of period τ satisfying $\|h_1\|_{H^{\frac{1}{3}}(0,\tau)} \leq \delta$, then (4.3) admits a time periodic solution*

$$u^* \in C_b(0, \infty; L^2(0, L)) \cap L^2_{loc}(0, \infty; H^1(0, L)),$$

which is locally exponentially stable.

The same result holds for following system associated to the IBVP (4.1):

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & x \in (0, L),\ t > 0, \\ u(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t). \end{cases} \tag{4.4}$$

In fact, using the same approach as that in [33] with a slight modification, we have the following theorem for the system (4.4).

Theorem 4.1. *There exists a $T > 0$ and $\delta > 0$ such that if $\vec{h} \in B_T$ is a time -periodic function of period τ satisfying*

$$\|\vec{h}\|_{B_T} \leq \delta,$$

then system (4.4) admits a admits a time periodic solution

$$u^* \in Y_T,$$

which is locally exponentially stable.

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