A short proof for interval tournaments

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Abstract

We give a short proof of the following result of Brown, Busch and Lundgren: If a tournament $T$ of order $n$ is an interval tournament, then $T$ contains a transitive subtournament of order $n - 1$.

For all background and definitions, the reader is referred to [1]. One of the two main results of [1] is the following characterization of interval tournaments.

**Theorem 1.** A tournament $T$ of order $n$ is an interval tournament if and only if $T$ contains a transitive subtournament of order $n - 1$.

Sufficiency is easily shown by constructing a zero partition of $A(T)$ when $T$ contains the required transitive subtournament (see Lemma 3.2 in [1]). This supplemental note gives a short proof of the converse, using some basic characteristics of zero partitions.

**Acyclic vertex partitions of loopless interval digraphs**

Let $D$ be a loopless interval digraph with permuted and zero-partitioned adjacency matrix $A(D)$. For each vertex $v$ let $r(v)$ and $c(v)$ be the row and column of $A(D)$ corresponding to $v$. Since $D$ is loopless, $a_{r(v),c(v)} = 0$ and this entry is either in $C$ or in $R$. Let $V_r = \{v \mid a_{r(v),c(v)} \in R\}$ and $V_c = \{v \mid a_{r(v),c(v)} \in C\}$. Note that a given $A(D)$ may have more than one zero partition, and hence $V_r$ and $V_c$ are defined with respect to a particular zero partition.

**Observation 2.** Every subset of $V_r$ or of $V_c$ induces an acyclic subgraph of $D$.

**Proof.** Let $S \subseteq V_r$. For $|S| = 1$, the result is immediate. Now let $|S| = m$ and assume the result holds for sets of size $m - 1$. Note that the submatrix associated with $S$ is an $m \times m$ matrix which contains an $R$ in every column. Thus, this submatrix contains an $R$ in column one, and the row containing this $R$ must consist of all $Rs$. This row corresponds to a vertex $v$ with out-degree zero in the digraph induced on $S$, and so this vertex is not in any cycle of the digraph induced on $S$. It then follows by induction that $S - \{v\}$ is acyclic, and hence so is $S$.\qed
Corollary 3. The vertex set of every interval tournament can be partitioned into two transitive subtournaments.

This corollary can be used in many cases to prove that a given tournament is not an interval tournament. As just one example, the Paley tournament (or quadratic residue tournament) of order seven does not contain a transitive subtournament of order four, and thus does not admit a vertex partition into two transitive subtournaments (see [2]).

Next, we show that the unique (up to isomorphism) strong four tournament does not have a zero partition with $|V_r| = |V_c|$.

Observation 4. No zero partition of the strong four tournament has $|V_c| = 2$.

Proof. Assume that such a tournament has a zero partition such that $|V_c| = 2$. We use the fact that no $2 \times 2$ submatrix of a zero partitionable matrix can contain exactly two $R$’s or exactly two $C$’s unless they share a row or a column.

The (unpartitioned) adjacency matrix of the unique strong four tournament is:

\[
\begin{array}{ccccc}
  & v_1 & v_2 & v_3 & v_4 \\
v_1 & 0 & 1 & 1 & 0 \\
v_2 & 0 & 0 & 1 & 1 \\
v_3 & 0 & 0 & 0 & 1 \\
v_4 & 1 & 0 & 0 & 0 \\
\end{array}
\]

If the zero in row one, column four is labeled $R$, then both the zero in row two and column two and the zero in row three and column three be labeled $C$; otherwise we have a $2 \times 2$ submatrix containing exactly two $R$’s which are not in the same row or column. So $\{v_2, v_3\} \in V_c$ and since $|V_c| = 2$ by assumption, $V_r = \{v_1, v_4\}$. Thus, we have the following partial labeling of our zero partition.

\[
\begin{array}{ccccc}
  & v_1 & v_2 & v_3 & v_4 \\
v_1 & R & 1 & 1 & R \\
v_2 & 0 & C & 1 & 1 \\
v_3 & 0 & 0 & C & 1 \\
v_4 & 1 & 0 & 0 & R \\
\end{array}
\]

Now observe that the entries shown in blue require that the blue 0 be in $C$. Similarly, the blue one and the three red entries require that the red 0 be in $C$ as well. But now we have a $2 \times 2$ submatrix (rows two and four and columns one and three) with exactly two $C$’s which appear in distinct rows and columns. This contradiction shows that either $v_1 \in V_c$ or $v_4 \in V_c$ and hence any partition of the adjacency matrix of the strong tournament of order four has $|V_c| = 3$. If the zero in row one, column four is labeled $C$, then an identical argument with the roles of $R$ and $C$ reversed shows that $|V_r| = 3$. Thus, no zero partition exists with $|V_r| = |V_c| = 2$. \qed
**Characterization of interval tournaments**

We can now complete the proof of Theorem 1 for all strong tournaments with the following lemma.

**Lemma 5.** If $T$ is a strong interval tournament of order $n$, then $T$ contains a transitive subtournament or order $n - 1$.

**Proof.** Let $T$ be an interval tournament, with a zero partition of $A(T)$ which determines sets $V_c = \{y_0, y_1, \ldots, y_a\}$ and $V_r = \{x_0, x_1, \ldots, x_b\}$ indexed in the transitive order (so $x_j \to x_i$ and $y_j \to y_i$ if and only if $j > i$).

First, assume without loss of generality that $x_0 \to y_0$. If $a > 1$ and $b > 1$, then there is a path from $y_0$ to a vertex in $V_r - \{y_0\}$ since $T$ is strong. Choose a shortest such path $P$. Note that all internal vertices of $P$ are in $V_c$, and since this set forms a transitive subtournament, there at most two such internal vertices and $P$ has length two or three. If $P$ has length three, then the four vertices of this path form a strong subtournament consisting of two vertices from each of $V_c$ and $V_r$. On the other hand if the path has length two (say, $y_0x_jy_i$), then either $y_i \to x_0$ and $y_0x_jy_iy_0x_0y_0$ is a cycle of length four, or $x_0 \to y_i$ and $y_0x_jx_0y_iy_0$ is a cycle of length four. In either case, these four vertices, two each from $V_c$ and $V_r$, induce a strong subtournament. Having arrived at a contradiction, we conclude that either $a \leq 1$ or $b \leq 1$, and hence $T$ has a transitive subtournament of order $n - 1$. 

To complete the characterization of interval tournaments, we now must show the above result holds for tournaments that are not strongly connected. For tournaments $T$ with a single non-trivial strong component of order $k < n$, this is routine. If $T$ is an interval tournament, then so is the non-trivial strong component, and by the theorem above, this subtournament has a transitive subtournament of order $k - 1$. This subtournament, combined with the trivial strong components of $T$, gives a transitive subtournament of order $(k - 1) + (n - k) = n - 1$. Finally, as noted in [1] (just before Theorem 1.3), no tournament with two or more non-trivial strong components is an interval tournament, since every such tournament contains a subtournament which corresponds to the bipartite graph known as “the insect” which is not an interval bigraph (see [3]).

**References**

