2 Combinatorics – Counting Techniques

To define a probability function $P$ over a sample space $S$, it is necessary to know the probability of occurrence of any subset of $S$. When $S$ is finite and the probability of each outcome of $S$ is known, then we can compute the probability of any subset of $S$. In the special case that $S$ is finite and has equally likely outcomes; i.e., that

$$S = \{x_1, x_2, \ldots, x_n\} \text{ and } P(x_i) = \frac{1}{n}, \ i = 1, \ldots, n,$$

for an event $A$ that contains $m$ distinct outcomes,

$$P(A) = \sum_{i=1}^{m} \frac{1}{n} = \frac{m}{n} = \frac{\text{size}(A)}{\text{size}(S)}.$$

Combinatorics techniques provide effective methods for counting the size of events.

2.1 Definitions

**Definition 7** An arrangement of $n$ symbols in a definite order is called a permutation of the $n$ symbols.

The number of permutations of $n$ items taken $n$ at a time is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

By convention, $0! = 1$. The number of permutations of $n$ items taken $r$ at a time is

$$nPr = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1).$$

**Definition 8** The number of subsets of size $r$ that can be formed from a set with $n$ elements is called a combination of $n$ things taken $r$ at a time, and is denoted by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{nPr}{r!}.$$

Consider the set $\{1,2,3\}$. The number of permutations on this set taken one at a time is

$$3P_1 = 3, \text{ and they are: } 1, 2, 3.$$

The number of combinations on this set taken 1 at a time is

$$\binom{3}{1} = 3, \text{ and they are: } 1, 2, 3.$$
The number of permutations on this set taken two at a time is
\(3P_2 = 6\), and they are: \((1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\).

The number of combinations on this set taken two at a time is
\(\binom{3}{2} = 3\), and they are: \(\{1, 2\}, \{1, 3\}, \{2, 3\}\).

### 2.2 Sampling without replacement: Ordered vs. unordered sample spaces

Many combinatorics problems can be thought of in terms of drawing balls from a jar. In the case that every ball drawn stays out of the jar once drawn, we say the balls are drawn without replacement. For any experiment that corresponds to this setup, when calculating probabilities of event occurrences, it is equivalent to use an ordered or an unordered sample space, provided consistency is maintained. In other words, if the sample space is (un)ordered, the event must be (un)ordered as well. The following two examples illustrate this.

#### 2.2.1 Card example

Two cards are drawn from a pack of cards without replacement. Calculate the probability 0 2’s, 1 2, and 2 2’s are drawn.

**Ordered sample space:**

\[
P(0 \ 2's) = \frac{48 \times 47}{52 \times 51} = 0.8507 \\
P(1 \ 2) = \frac{4 \times 48 + 48 \times 4}{52 \times 51} = 0.1448 \\
P(2 \ 2's) = \frac{4 \times 3}{52 \times 51} = 0.0045.
\]

**Unordered sample space:**

\[
P(0 \ 2's) = \frac{48}{\binom{52}{2}} = 0.8507 \\
P(1 \ 2) = \frac{\binom{4}{1} \times \binom{48}{1}}{\binom{52}{2}} = 0.1448 \\
P(2 \ 2's) = \frac{4}{\binom{52}{2}} = 0.0045.
\]

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2.2.2 Manufacturing example

A production facility manufactures 5 types of parts for 3 different companies, X, Y, and Z. Each part is identified by its type and the company for which it is destined (e.g., part 3X is a type 3 part manufactured for company X). A single production run consists of one part of each type for each company: 1X, . . . , 5X, 1Y, . . . , 5Y, 1Z, . . . , 5Z. The total number of parts in one production run is 15. Suppose 3 parts are chosen at random from a given production run to inspect for faults. Calculate the following: (1) the probability that all 3 parts are the same type, (2) the probability that all 3 parts are different types, (3) the probability that all 3 parts are from the same company, and (4) the probability that all 3 parts are from different companies.

Ordered sample space:

\[
P(\text{same type}) = \frac{3! \times 5}{15 \times 14 \times 13} = 0.0110
\]

\[
P(\text{different types}) = \frac{15 \times 12 \times 9}{15 \times 14 \times 13} = 0.5934
\]

\[
P(\text{same company}) = \frac{5 \times 3}{15 \times 14 \times 13} = 0.0659
\]

\[
P(\text{different companies}) = \frac{15 \times 10 \times 5}{15 \times 14 \times 13} = 0.2747
\]

Unordered sample space:

\[
P(\text{same type}) = \frac{5}{\binom{15}{3}} = 0.0110
\]

\[
P(\text{different types}) = \frac{\binom{5}{3} \times 3 \times 3 \times 3}{\binom{15}{3}} = 0.5934
\]

\[
P(\text{same company}) = \frac{3 \times \binom{5}{3}}{\binom{15}{3}} = 0.0659
\]

\[
P(\text{different companies}) = \frac{5 \times 5 \times 5}{\binom{15}{3}} = 0.2747
\]
2.3 Sampling with replacement: Ordered vs. unordered sample spaces

Balls may also be drawn from a jar such that each ball is returned to the jar after it is drawn. In this case, we say the balls are drawn with replacement. When the order in which the balls are drawn from the jar is recorded, the number of ways \( r \) balls may be drawn from a jar containing \( n \) balls is \( n^r \). Otherwise, when the order is not recorded, the number of ways \( r \) balls may be drawn from a jar containing \( n \) balls is \( \binom{n+r-1}{r} \). When deciding whether or not to order the sample space, caution is warranted, as the following simple example illustrates.

Suppose two balls are drawn with replacement from a jar containing 1 red ball, 1 white ball, and 1 black ball. If order is important, the possible samples are

\[
\{(B, W), (W, B), (B, R), (R, B), (W, R), (R, W), (B, W), (W, B), (R, R)\},
\]

and the total number of samples is \( 3^2 = 9 \). If order is not important, the possible samples are

\[
\{(B, W), \{B, R\}, \{W, R\}, \{B, B\}, \{W, W\}, \{R, R\}\},
\]

and the total number of samples is \( \binom{3+2-1}{2} = 6 \).

Now, consider the probability of drawing 2 white balls. It first appears that if the sample space is ordered, this probability is \( \frac{1}{9} \), but if it is unordered, the probability is \( \frac{1}{6} \). What is wrong with this reasoning, and what is the correct probability?

2.4 Example problems

Most all combinatorics problems can be thought about either in terms of drawing balls from jars or in terms of distributing balls into jars. The following chart is useful in deciding the appropriate formula for a given situation.

<table>
<thead>
<tr>
<th>Sample drawn</th>
<th>Ordered</th>
<th>Unordered</th>
</tr>
</thead>
<tbody>
<tr>
<td>w.o.r. ( n^r )</td>
<td>( \binom{n}{r} )</td>
<td>w.e.</td>
</tr>
<tr>
<td>w.r. ( n^r )</td>
<td>( \binom{n+r-1}{r} )</td>
<td>w.o.e.</td>
</tr>
</tbody>
</table>

Distinguishable balls | Indistinguishable balls | Balls distributed

- The number of ways to draw samples of size \( r \) from a jar containing \( n \) distinguishable balls.

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2.4.1 Problem 1: Derivation of the binomial theorem

The binomial theorem states that for any real numbers \( x \) and \( y \), and for any integer \( n \geq 0 \),

\[
(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.
\]

To see this, for \( i = 0, \ldots, n \), consider the following

\[
\frac{(x + y)^n}{x^{n-i}} = \frac{(x + y)(x + y) \cdots (x + y)}{x^{i} \cdot \cdots}.
\]

For a specified \( i \), to arrive at the term \( x^i y^{n-i} \), each of the blanks may contain either an \( x \) or a \( y \). Hence the number of terms having an \( x^i y^{n-i} \) is exactly the number of ways in which \( i \) “balls” can be placed in \( n \) “distinct jars”. Note that for a given \( n \) the total number of terms is \( 2^n \).

2.4.2 Problem 2: Poker hands

5 cards are drawn from a deck of 52 cards. The following calculates the probabilities of different hands.

\[
P(4 \text{ of a kind }) = \frac{13 \times 48}{\binom{52}{5}}
\]

\[
P(\text{ exactly 1 pair of aces }) = \frac{\left( \binom{4}{2} \times \binom{12}{3} \right) \times 4^3}{\binom{52}{5}}
\]

\[
P(\text{ 2 pairs }) = \frac{\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times \binom{11}{1} \times \binom{4}{1}}{\binom{52}{5}}
\]

\[
P(\text{ a flush }) = \frac{\binom{4}{1} \times \binom{13}{5}}{\binom{52}{5}}
\]

\[
P(\text{ a straight }) = \frac{4^5 \times 10}{\binom{52}{5}}
\]

When calculating the probability that exactly 1 pair of aces is obtained, why is the numerator not

\[
\binom{4}{2} \times \binom{50}{3}?
\]
2.4.3 Problem 3: Distributing balls into jars

There are 5 offices and 12 identical desks. How many ways can the 12 desks be distributed in the 5 offices?

\[
\binom{5 + 12 - 1}{12} = \binom{16}{12} = 1820
\]

How many ways can the 12 desks be distributed in the 5 offices if we require that every office has 2 desks?

\[
\binom{5 + 2 - 1}{2} = \binom{6}{2} = 15
\]

How many ways can the 12 desks be distributed in the 5 offices if we require that every office has 2 desks and no office has more than 3 desks?

\[
\binom{5}{2} = 10
\]

2.4.4 Problem 4: Arranging objects in a circle

(This is problem 1.7.11 in Hayter.) In an arrangement of \( n \) objects in a circle, an object’s neighbors are important, but an object’s place in the circle is not important. So rotations of a given arrangement are considered to be the same arrangement. How many different arrangements are there?

\[
\frac{n!}{n} = (n - 1)!
\]

2.4.5 Problem 5: Arranging objects in a line

(This is problem 1.7.13 in Hayter.) In how many ways can 6 people sit in 6 seats in a line in a cinema if we require that Andrea sit next to Scott?

\[
5! \times 2! = 240
\]

What if Andrea refuses to sit next to Scott?

\[
6! - 240 = 480
\]