ABSTRACT
This paper presents an analysis to create a general singularity condition for a mechanism that contains a deformable closed contour. This kinematic architecture is widely used in rigid-body shape changing mechanisms. The general singularity equation is reduced to a condensed form, which allows geometric relationships to be readily detected. A method for formulating the singularity condition for a mechanism with N links in the closed contour, knowing the condition for the N – 1 mechanism, is also given.

1 INTRODUCTION
During the operation of a mechanism, a configuration can arise such that the input link is no longer able to move the mechanism. The mechanism becomes stationary, and the mechanical advantage reduces to zero [1]. This stationary position of a mechanism is termed a singular configuration and must be avoided when attempting to operate the mechanism with a single input crank. Mathematically, a singular position occurs when the matrix that relates the input speeds with the output speeds becomes rank deficient, as discussed by Gosselin [2]. Singularity analysis of a particular mechanism is the study of the conditions that lead to the stationary configuration.

A recent promising application of closed loop linkages is shape-changing rigid-body mechanisms [3]. Planar linkages, comprising a closed chain of N rigid links connected with revolute joints, can be designed to approximate a shape change defined by a set of “morphing curves”. The value N is selected by balancing the approximation to the desired profiles, which is generally better with a larger N, and the complexity of the mechanism, which is generally reduced with a smaller N [4]. A final shape-changing mechanism may be designed to constrain the rigid links to a single-degree-of-freedom linkage. Of the N links forming the closed loop, one is binary, while N – 1 are ternary links. The other revolute joints on the ternary links connect constraining dyads. A mechanism with six links (N = 6) in its shape changing, closed contour is shown in Fig. 1. Also the mechanism contains five ternary links and five constraining dyads (N – 1 = 5).

A classical approach for describing planar mechanism composition is with Assur groups. An Assur Kinematic Chain (AKC) is defined as the minimum arrangement of n links that creates a rigid structure [6]. That is, another rigid structure cannot be created by suppressing one or more links of an AKC. An AKC can be converted into a single-degree-of-freedom mechanism by replacing one link in the structure by one pair of links. The mechanism shown in Fig. 1 is an Assur mechanism. If a dyad were removed, as indicated by the gray grounded pivot, an AKC structure would result. It is not possible to suppress additional links and generate another structure.

Replacing a link of an existing mechanism with an AKC does not alter the degrees of freedom of the new mechanism. Thus, mechanisms of great complexity can be constructed by the sequential addition of AKCs to simpler chains [9]. The usefulness of AKCs arise in position analysis. Since an AKC is a structure, constraint equations can be readily generated to identify the position of all links in the chain. This approach allows a mod-
ularization approach to position analysis. Complex linkages can be analyzed by dividing them into a sequence of groups [10]. The class of an AKC is commonly referred to by the number of links contained in a deformable closed contour [8]. The accepted nomenclature is to specify the class with a Roman numeral. The order of an AKC refers to the number of external kinematic joints. Order, which is sometimes referred to as series, can also be defined as the number binary links in the closed chain that are replaced with ternary, or quaternary links [7]. For example, if an Assur Class IV AKC has two of the four links that comprise the closed chain as ternary it is referred to as order 2. This Assur mechanism is denoted as IV/2.

The closed-loop, shape changing mechanisms described earlier are Assur $N$, Order $N - 1$. The Class arises because $N$ links are incorporated in the the shape-changing, closed chain. The Order is because $N - 1$ dyads are used to constrain the mechanism. The Order is also identified with $N - 1$ ternary links included in the shape-changing loop. A general model of the Assur $VI/5$ is illustrated in Figure 1.

Formulation of singularities for shape changing mechanisms is necessary to ensure that the linkage can be driven between a set of “morphing curves”. The singularity configurations in these complex linkages are seldom intuitive. For example, the Assur IV/3 shown in Fig 2 is at a singularity. Note that no links are parallel or perpendicular. Further, lines along the links are extended to illustrate that no other special relationships exist, such as three lines intersecting at a point. This paper presents an method to create a concise singularity equation for an Assur mechanism that contains a deformable closed contour. In designing shape-changing mechanisms of Assur Class $N$, Order $N - 1$, the authors have experienced significant difficulty in locating an input link that rotates monotonically to drive the morphing among the desired shapes because the singular configurations are not intuitively identifiable. Generating the most concise statement of the singularities enables the designer to identify, and thus avoid, the special geometrical relationships that lead to singularities. This avoidance dramatically reduces the design space to be searched, speeding the synthesis process and enabling design optimization based upon more advanced criteria than simply singularity avoidance, such as force transmission. The remainder of the paper is organized as follows. The singularity condition is generated for a mechanism with three links in the closed contour in Section 2. Section 3 expands the analysis to an Assur IV/3 mechanism, while Section 4 deals with an Assur V/4. Section 5 utilizes these results to generalize the singularity condition for an Assur $N/N - 1$.

2 Specialization to a Mechanism with Three Links in the Closed Loop

A mechanism with three links defining the closed-loop portion is shown in Fig. 3. The mechanism is composed of 6 moving links, 7 revolute joints, 2 ternary links and has a single degree of freedom. A closed kinematic chain of mobility 0 is obtained by suppressing one dyad, $a_1$. However, the resulting structure is not
an AKC since additional links \((g_2\) and \(a_2\)) can be subsequently suppressed to form another closed kinematic chain of mobility \(0\). Still, the mechanism is analogous to an Assur III/2.

It is noted that the mechanism in Fig. 3 is merely a four-bar linkage since the three link closed-loop forms a rigid structure. This trivial case is not a shape-changing mechanism, but is explored to develop equations for a singularity analysis which will provide a starting point for further generalizations.

The mechanism has 11 physical parameters \((a_i, \ b_i, \ d_i, \ e_i, \ i = 1,2, \text{ and } f, \ g_2x, \ g_2y)\) and 7 joint variables \((\theta_1, \gamma_1, \theta_2, \phi_2, \psi_2, \beta)\). The loop closure equations for the mechanism are

\[
\begin{align*}
\theta_1 &+ e_1 \cos \gamma_1 - g_2x - a_2 \cos \theta_2 - b_2 \sin \phi_2 = 0, \\
\theta_1 &+ e_1 \sin \gamma_1 - g_2y - a_2 \sin \theta_2 - b_2 \cos \phi_2 = 0, \\
d_1 \cos \psi_1 + d_2 \cos \psi_2 + f \cos \beta = 0, \\
d_1 \sin \psi_1 + d_2 \sin \psi_2 + f \sin \beta = 0, \\
\gamma_1 - \psi_1 - C_1 = 0, & \quad \phi_2 - \psi_2 - C_2 = 0,
\end{align*}
\]

where \(C_1\) and \(C_2\) are due to the rigid ternary links, noting that \(2b_2e_1 \cos C_1 = b_1^2 + e_1^2 - d_1^2\) and \(2b_2d_2 \cos C_2 = e_2^2 - b_2^2 - d_2^2\). The time derivative of the loop equations relates the velocities of the joint parameters, which is important in identifying the singular conditions. Being linear, the velocity equations can be written as

\[
\begin{align*}
\ddot{\theta}_1 + M_3 \ddot{\psi}_3 &= 0,
\end{align*}
\]

where

\[
\ddot{\psi}_3 = \left\{ \begin{array}{c} \ddot{\gamma}_1, \ddot{\theta}_2, \ddot{\phi}_2, \ddot{\psi}_2, \ddot{\beta} \end{array} \right\}^T
\]

The nomenclature used in Eqs. 3 and 4 and continued through the remainder of the paper is as follows:

\[
A_{is} = a_i \sin \theta_i, \quad A_{ic} = a_i \cos \theta_i, \quad B_{is} = b_i \sin \phi_i, \\
B_{ic} = b_i \cos \phi_i, \quad D_{is} = d_i \sin \psi_i, \quad D_{ic} = d_i \cos \psi_i,
\]

for \(1 \leq i \leq N - 1\), and

\[
F_s = f \sin \beta, \quad F_c = f \cos \beta.
\]

When driving the mechanism with \(\theta_1\), a singularity is encountered when the matrix \(M_3\) in Eq. 2 loses rank,

\[
\det[M_3] = 0.
\]

Taking the determinant generates the singularity equation

\[
\begin{align*}
A_{2i}B_{2i}D_{ic}F_c - A_{2i}B_{2i}D_{ic}F_c - A_{2i}B_{2i}D_{ic}F_c \\
+ A_{2i}B_{1ic}F_c + A_{2i}D_{ic}E_{1ic}F_c - A_{2i}D_{ic}E_{1ic}F_c \\
- A_{2i}D_{ic}E_{1ic}F_c + A_{2i}D_{ic}E_{1ic}F_c = 0.
\end{align*}
\]

Fully expanded, Eq. 9 contains 8 terms. Experience with trigonometric substitutions lead the authors to simplify the singularity equation to a concise form with 2 terms:

\[
\begin{align*}
d_1 b_2 \sin(\phi_2 - \theta_2) \sin(\psi_1 - \beta) \\
+ e_1 d_2 \sin(\gamma_1 - \theta_2) \sin(\psi_2 - \beta) &= 0.
\end{align*}
\]
The loop closure equations for this mechanism are

\[
\begin{align*}
&\theta_1 \cos \theta_1 + e_1 \cos \gamma_1 - g_{2x} - a_2 \cos \theta_2 - b_2 \cos \phi_2 = 0, \\
&\theta_1 \sin \theta_1 + e_1 \sin \gamma_1 - g_{2y} - a_2 \sin \theta_2 - b_2 \sin \phi_2 = 0, \\
&a_2 \cos \theta_2 + e_2 \cos \gamma_2 - g_{3x} - a_3 \cos \theta_3 - b_3 \cos \phi_3 = 0, \\
&a_2 \sin \theta_2 + e_2 \sin \gamma_2 - g_{3y} - a_3 \sin \theta_3 - b_3 \sin \phi_3 = 0, \\
&\gamma_1 - \psi_1 - C_1 = 0, \\
&\phi_2 - \psi_2 - C_2 = 0, \\
&\gamma_2 - \psi_2 - C_3 = 0, \\
&\phi_1 - \psi_3 - C_4 = 0.
\end{align*}
\]

In addition to \( C_1 \) and \( C_2 \) defined in the previous case, constants \( C_3 \) and \( C_4 \) are used in the loop equations to insure that the ternary links remain rigid. Again, the constants are derivable from the physical parameters, \( 2d_2b_2c_3 = d_2^2 + e_2^2 - b_2^2 \) and \( 2b_3d_3c_4 = e_3^2 - b_3^2 - d_3^2 \). Taking the time derivative of the loop equations and arranging into matrix form gives

\[
\vec{K}_4 \vec{\theta}_1 + M_4 \vec{\dot{v}}_4 = 0,
\]

where

\[
\vec{K}_4 = \{-A_{1s}, A_{1c}, 0, 0, 0, 0, 0, 0, 0\}^T,
\]

\[
M_4 = \begin{bmatrix}
-E_{1s} & -A_{2s} & B_{2s} & 0 & 0 & 0 & 0 & 0 & 0 \\
E_{1c} & A_{2c} & B_{2c} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -A_{2s} & 0 & -E_{2s} & A_{3s} & B_{3s} & 0 & 0 & 0 \\
0 & -A_{2c} & 0 & E_{2c} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -D_{1s} & -D_{2s} & -D_{3s} \\
0 & 0 & 0 & 0 & 0 & 0 & D_{1c} & D_{2c} & D_{3c} \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0
\end{bmatrix},
\]

and

\[
\vec{\dot{v}}_4 = \begin{bmatrix}
\gamma_1, \dot{\theta}_2, \phi_2, \gamma_2, \dot{\theta}_3, \phi_3, \gamma_3, \psi_1, \psi_2, \psi_3, \phi_3
\end{bmatrix}^T.
\]

When driving the mechanism with \( \theta_1 \), a singularity will be encountered when

\[
\det[M_4] = 0.
\]
Taking the determinant generates the singularity equation

\[
A_2 A_3 F_4 E_1 D_1 D_2 B_3 + A_2 A_3 F_4 E_1 D_2 B_3 - A_2 A_3 F_4 E_1 D_2 B_3 - A_2 A_3 F_4 E_1 D_2 B_3 \\
+ A_2 A_3 F_4 E_1 D_2 B_3 + A_2 A_3 F_4 E_1 D_2 B_3 \\
- A_2 A_3 F_4 E_1 D_2 B_3 - A_2 A_3 F_4 E_1 D_2 B_3 \\
+ A_2 A_3 F_4 E_1 D_2 B_3 + A_2 A_3 F_4 E_1 D_2 B_3 \\
- A_2 A_3 F_4 E_1 D_2 B_3 - A_2 A_3 F_4 E_1 D_2 B_3 \\
+ A_2 A_3 F_4 E_1 D_2 B_3 + A_2 A_3 F_4 E_1 D_2 B_3 \\
- A_2 A_3 F_4 E_1 D_2 B_3 - A_2 A_3 F_4 E_1 D_2 B_3 \\
+ A_2 A_3 F_4 E_1 D_2 B_3 + A_2 A_3 F_4 E_1 D_2 B_3 \\
- A_2 A_3 F_4 E_1 D_2 B_3 - A_2 A_3 F_4 E_1 D_2 B_3 = 0.
\]  

(16)

\[d_1 b_2 b_3 \sin(\phi_2 - \phi_2) \sin(\phi_3 - \phi_3) \sin(\psi_1 - \beta) + e_1 d_1 b_3 \sin(\gamma_1 - \phi_2) \sin(\phi_3 - \phi_3) \sin(\psi_2 - \beta) + e_1 b_2 d_3 \sin(\gamma_1 - \phi_2) \sin(\gamma_2 - \phi_3) \sin(\psi_3 - \beta) = 0.
\]  

(17)

The 32 terms in Eq. 17 have been reduced to 4 terms in Eq. 18. Again, the geometrical relationships can be more readily detected with the reduced version. For example, a singularity exists if links \(e_1, d_2, a_3,\) and \(b_3\) are parallel, giving \(\gamma_1 = \phi_2(\pm \pi) = \phi_3(\pm \pi)\). Note, however, that this results in three additional constraint equations on a single-degree-of-freedom mechanism. Therefore, this case will not arbitrarily exist as two of the physical parameters must be consistent with the loop equations given in Eq. 12.

With a judicious change to the definition of the variables, these results can be utilized for mechanisms driven with the other outer dyad, \(a_3\).

4 Specialization to a Mechanism with Five-Link Closed Loop and Four Ternary Links: Assur V/4

A mechanism with five links defining the closed-loop, shape-changing portion is shown in Fig. 5. This is categorized as an Assur V/4 with 9 moving links, 13 revolute joints, 4 ternary links and a single degree of freedom. Completing an analysis identical to those that led to Eqs. 10 and 18, the following singularity condition applied to the Assur V/4.

\[
d_1 b_2 b_3 b_4 \sin(\phi_2 - \phi_2) \sin(\phi_3 - \phi_3) \sin(\phi_4 - \phi_4) \sin(\psi_1 - \beta) + e_1 d_1 b_3 \sin(\phi_1 - \phi_2) \sin(\phi_4 - \phi_4) \sin(\psi_2 - \beta) + e_1 d_3 b_4 \sin(\phi_1 - \phi_2) \sin(\phi_4 - \phi_4) \sin(\psi_3 - \beta) + e_1 e_2 b_3 \sin(\phi_1 - \phi_2) \sin(\phi_4 - \phi_4) \sin(\psi_4 - \beta) + e_1 e_2 d_3 \sin(\phi_1 - \phi_2) \sin(\phi_3 - \phi_3) \sin(\psi_4 - \beta) + e_1 e_2 e_3 d_4 \sin(\phi_1 - \phi_2) \sin(\phi_3 - \phi_3) \sin(\psi_4 - \beta) + e_1 e_2 e_3 d_4 \sin(\phi_1 - \phi_2) \sin(\phi_3 - \phi_3) \sin(\psi_4 - \beta) = 0.
\]  

(18)

The general equation of 64 terms has been reduced to the 8 terms of Eq. 18. By judiciously redefining the parameters, the condition can be utilized for mechanisms driven with the other outer dyad, \(a_3\).

5 Generalization to a Mechanism with an \(N\) Link Closed-Loop and \(N-1\) Ternary Links: Assur \(N/N-1\)

When extending the Assur Class of the mechanism it becomes apparent that the new singularity condition builds upon the condition associated with the lower class mechanism. A series of rules were formulated that allows construction of the higher class mechanism. A general model of the Assur \(N/N-1\) is illustrated in Fig. 6. The singularity condition for Class
The final equation will contain $2^{N-2}$ terms.

6 Conclusions

This paper presents an analysis to create the singularity condition of a closed-loop, Assur mechanism. The general singularity equation was reduced to a condensed form, which allows geometric relationships to be readily detected. The identification of singularities of this extensible kinematic architecture is particularly relevant to rigid-link, shape-changing mechanisms. A procedure for generating the reduced form of the singularity condition for an Assur Class $N$ mechanism, knowing the condition for a Class $N-1$ is presented. Additionally, a procedure for producing the singularity condition, when the driving dyad is presented.

REFERENCES