Find the volume of the solid $S$ given in each of problems 1–6. If a technique for finding the volumes is specified, you must use that technique. If no technique is specified, you may use any legitimate technique.

1. Let $R$ be the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 4$. $S$ is the solid obtained by rotating $R$ about the line $y = 3$. Use disks or washers.

Solution: Because the axis of rotation is parallel to the $x$-axis and we are using disks or washers, everything will done in terms of $x$. Since $y = \sqrt{x}$ meets the $x$-axis ($y = 0$) at $x = 0$ we have $0 \leq x \leq 4$. Also, $y = \sqrt{x}$ meets $x = 4$ when $y = 2$ so the axis of rotation is above $R$. The large radius of rotation is $3 - 0 = 3$ and the small radius of rotation is $3 - \sqrt{x}$.

$$
Volume = \pi \int_0^4 [3^2 - (3 - \sqrt{x})^2] \, dx
$$

$$
= \pi \int_0^4 [9 - (9 - 6\sqrt{x} + x)] \, dx
$$

$$
= \pi \int_0^4 (6\sqrt{x} - x) \, dx
$$

$$
= \pi \left[ 4x^{3/2} - \frac{1}{2}x^2 \right]_0^4
$$

$$
= \pi \left[ 4(8) - \frac{1}{2}(16) \right]
$$

$$
= \pi (32 - 8)
$$

$$
= 24\pi
$$
2. Let $R$ be the region bounded by the graphs of $y = e^x$, $y = 0$, $x = 0$, and $x = 2$. $S$ is obtained by rotating $R$ about the line $x = 4$.

Solution: The axis of rotation is parallel to the $y$-axis. If we use disks or washers we would have to transform the functions into functions of $y$ rather than $x$, and we would have to use two integrals. So we will use shells for this volume. Everything will be done in terms of $x$. We have $0 \leq x \leq 2$. The radius of rotation is $4 - x$, and the height of a shell is $e^x - 0 = e^x$.

$$\text{Volume} = 2\pi \int_0^2 (4 - x)e^x \, dx$$
$$= 2\pi \int_0^2 (4e^x - xe^x) \, dx$$
$$= 2\pi \left[ (4e^x - xe^x) \right]_0^2$$

Use the formula $\int xe^x \, dx = xe^x - e^x$.

$$= 2\pi (4e^2 - 2e^2 - 0 + e^2 - 1)$$
$$= 2\pi (3e^2 - 5)$$

3. $R$ is the region bounded by the graphs of $x = y^3$, $x = 0$, and $y = 2$. $S$ is obtained by rotating $R$ about the line $y = -\frac{2}{5}$. Use shells.

Solution: The axis of rotation is parallel to the $x$-axis, and we are using shells, so everything will be done in terms of $y$. $R$ lies above the $x$-axis and the axis of rotation is below the $x$-axis, so the radius of rotation is $x - (-2/5) = x + 2/5$. The height of a shell is the distance from the $y$-axis to the curve $x = y^3$ or $y^3 - 0 = y^3$. The curve $x = y^3$ meets the $y$-axis when $y^3 = 0$ or $y = 0$, so $0 \leq y \leq 2$.

$$\text{Volume} = 2\pi \int_0^2 \left( y + \frac{2}{5} \right) y^3 \, dy$$
$$= 2\pi \int_0^2 \left( y^4 + \frac{2}{5}y^3 \right) \, dy$$
$$= 2\pi \left[ \frac{1}{5}y^5 + \frac{1}{10}y^4 \right]_0^2$$

$$= 2\pi \left( \frac{32}{5} + \frac{8}{5} \right)$$
$$= 2\pi \left( \frac{40}{5} \right)$$
$$= 16\pi$$
4. Let \( R \) be the region bounded by the graphs of \( x = y - y^2 \) and \( x = 0 \). \( S \) is obtained by rotating \( R \) about the \( y \)-axis.

Solution: The radius of rotation is parallel is the \( y \)-axis. If we were to use shells then the integral must be in terms of \( x \). The equation \( x = y - y^2 \) is much easier to do in terms of \( y \), so we will use disks. We won’t have washers because the axis of rotation is the same as a bound of \( R \). Everything will be in terms of \( y \). The curve \( x = y - y^2 \) meets the \( y \)-axis when \( y - y^2 = 0 \) so \( y = 0, 1 \). The radius of rotation is \( y - y^2 - 0 = y - y^2 \).

\[
\text{Volume} = \pi \int_{0}^{1} (y - y^2)^2 \, dy
\]
\[
= \pi \int_{0}^{1} (y^2 - 2y^3 + y^4) \, dy
\]
\[
= \pi \left( \frac{1}{3}y^3 - \frac{1}{2}y^4 + \frac{1}{5}y^5 \right) \bigg|_{0}^{1}
\]
\[
= \pi \left( \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right)
\]
\[
= \pi \left( \frac{10 - 15 + 6}{3} \right)
\]
\[
= \frac{\pi}{30}
\]
5. $S$ is a solid which intersects the $xy$-plane in the region $R$ bounded by the graphs of $x = y^2$ and $x = 1$. Every cross-section of $S$ perpendicular to the $x$-axis is an isosceles right triangle, one of whose legs spans $R$.

Solution: This is not a solid of rotation, so we must use slicing. The cross-sections are perpendicular to the $x$-axis so we will work in terms of $x$. The curve $x = y^2$ meets the $x$-axis when $x = 0$ so $0 \leq x \leq 1$. Also, when we solve $x = y^2$ for $y$ we get $y = \pm \sqrt{x}$. So the length of one leg of the triangle is $\sqrt{x} - (-\sqrt{x}) = 2\sqrt{x}$. The area of a cross section is $\frac{1}{2}(2\sqrt{x})^2 = 2x$. So the volume is given by the following integral.

$$\int_0^1 2x \, dx = x^2 \bigg|_0^1 = 1$$
6. Set up, but do not evaluate, an integral which gives the length of the curve \( y = \tan x \) for \( 0 \leq x \leq \pi/4 \).

Solution: We have \( y' = \sec^2 x \) and \( 1 + (y')^2 = 1 + \sec^4 x \). The arc length is \( \int_0^{\pi/4} \sqrt{1 + \sec^4 x} \, dx \).

7. Set up, but do not evaluate, an integral which gives the area of the surface obtained by rotating the curve \( y = \ln x \) (for \( 1 \leq x \leq 2 \)) about the \( x \)-axis.

Solution: We have \( y' = \frac{1}{x} \) and \( 1 + (y')^2 = 1 + \frac{1}{x^2} \). The surface area is \( 2\pi \int_1^2 \ln x \sqrt{1 + \frac{1}{x^2}} \, dx \).