1. Find the smallest value of $n$ for which we know that \[ \left| \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} - \sum_{k=1}^{n} \frac{(-1)^k}{k^2} \right| < .01. \]

Solution: We know that \[ \left| \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} - \sum_{k=1}^{n} \frac{(-1)^k}{k^2} \right| < a_{n+1}, \] so we need to find the first $n$ such that \[ a_{n+1} \leq .01. \]

\[
\frac{1}{(n+1)^2} \leq .01 \\
100 \leq (n+1)^2 \\
10 \leq n + 1 \\
9 \leq n
\]

So the smallest $n$ is 9.

2. Find the Maclaurin series for the function $f(x) = \int \cos(x^3) \, dx$.

Solution: \[ f(x) = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} \, dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{6n}}{(2n)!} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!(6n+1)} \]

3. Let $f(x) = \sum_{n=1}^{\infty} \frac{(3x - 2)^n}{n3^n}$. Find $\frac{df}{dx}$.

Solution:

\[
\frac{df}{dx} = \sum_{n=1}^{\infty} \frac{d}{dx} \left( \frac{(3x - 2)^n}{n3^n} \right) \\
= \sum_{n=1}^{\infty} n(3x - 2)^{n-1} \cdot \frac{3}{n3^n} \\
= \sum_{n=1}^{\infty} \frac{(3x - 2)^{n-1}}{3^{n-1}} \\
= \sum_{n=0}^{\infty} \frac{(3x - 2)^n}{3^n}
\]
4. Is the series \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \] absolutely convergent, conditionally convergent, or divergent? Support your answer.

Solution: First check for absolute convergence. The series \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \] is a divergent \( p \)-series (\( p = 1/2 \)), so \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \] is not absolutely convergent. But \( \frac{1}{\sqrt{n}} \) is a positive, decreasing sequence and \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \), so \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \] converges by the alternating series test. Therefore \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \] is conditionally convergent.

5. Let \( C \) be the curve given by \( x(t) = 6 \sin t \) and \( y(t) = t^2 + t \). Find the slope of the line tangent to \( C \) at the point \( (0,0) \).

Solution: If \( y = 0 \) then \( t = -1 \) or \( t = 0 \). But \( x(-1) \neq 0 \) and \( x(0) = 0 \) so we want \( t = 0 \).

\[
\frac{dx}{dt} = 6 \cos t \quad \frac{dy}{dt} = 2t + 1
\]

\[
\frac{dy}{dx} = \frac{2t + 1}{6 \cos t}
\]

\[
\left. \frac{dy}{dx} \right|_{t=0} = \frac{1}{6}
\]

6. Find parametric equations which describe the motion of an object which travels along a circle of radius 4 in such a way that it is located at the point \( (0,4) \) when \( t = 0 \) and travels clockwise halfway around the circle in one unit of time.

Solution: The usual parametrization of the circle is \( x(t) = \cos t \) and \( y(t) = \sin t \). The motion this describes is counterclockwise, so we will change it to \( x(t) = \sin t \) and \( y(t) = \cos t \). This gives us the desired clockwise motion. The radius of the circle thus described is 1, and we want a radius of 4, so we will change the parametrization to \( x(t) = 4 \sin t \) and \( y(t) = 4 \cos t \). Notice that \( x(0) = 0 \) and \( y(0) = 4 \), so the object will be at the point \( (0,4) \) when \( t = 0 \). Finally, an object following this parametrization will complete one circle in \( 2\pi \) units of time. We want the period to be 2 so that one-half of the circle is covered in one unit of time. We must have a parametrization of the form \( x(t) = 4 \sin \alpha t \) and \( y(t) = 4 \cos \alpha t \) for some value of \( \alpha \) which changes the period to 2. Then \( 2 = (2\pi)/\alpha \) so \( \alpha = (2\pi)/2 = \pi \). One possible parametrization is

\[
x(t) = 4 \sin \pi t \quad y(t) = 4 \cos \pi t
\]
7. Find the radius and interval of convergence of \( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)2^n} \).

Solution: Use the Ratio Test to determine the radius of convergence.

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+2)2^{n+1}} \cdot \frac{(n+1)2^n}{(-1)^n x^n} \right| = \lim_{n \to \infty} \frac{n+1}{n+2} \cdot \frac{1}{2} \cdot |x| = \frac{|x|}{2} < 1
\]

The series will converge when \(|x| < 2\) so the radius of convergence is 2. We must next determine whether the series converges at \(-2\) and 2.

\[
\sum_{n=0}^{\infty} \frac{(-1)^n (-3)^n}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{1}{n+1} \quad \text{which diverges}
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \quad \text{which converges}
\]

So the interval of convergence is \((-2, 2]\).

8. Find the Maclaurin series for the function \( f(x) = (1+x)e^{-x} \). Your answer should contain only one summation and one power of \( x \).

Solution: Use the fact that \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

\[
f(x) = (1+x) \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}
\]

\[
= (1+x) \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)!}
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)!}
\]

\[
= 1 + \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} \right) x^n
\]

\[
= 1 + \sum_{n=1}^{\infty} \left( \frac{1}{n!} - \frac{1}{(n-1)!} \right) x^n
\]

\[
= 1 + \sum_{n=1}^{\infty} \left( \frac{1-n}{n!} \right) x^n
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{(-1)^{n+1}(n-1)}{n!} \right) x^n
\]
9. Let \( f(x) = \ln(\cos x) \). Find \( P_4(x) \), the fourth order Taylor polynomial generated by \( f \) using \( a = 0 \).

Solution:

\[
\begin{align*}
\quad & f(x) = \ln(\cos x) & f(0) &= \ln 1 = 0 \\
\quad & f'(x) = -\frac{\sin x}{\cos x} = -\tan x & f'(0) &= 0 \\
\quad & f''(x) = -\sec^2 x & f''(0) &= -1 \\
\quad & f^{(3)}(x) = -\sec^2 x \tan x & f^{(3)}(0) &= 0 \\
\quad & f^{(4)}(x) = -4\sec^2 x \tan^2 x - 2\sec^4 x & f^{(4)}(0) &= -2
\end{align*}
\]

\[
P_4(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 = -\frac{1}{2} x^2 - \frac{2}{24} x^4 = -\frac{1}{2} x^2 - \frac{1}{12} x^4
\]

10. Let \( C \) be the curve given by the equations \( x(t) = -\frac{1}{\sqrt{t}} \) and \( y(t) = \frac{1}{t} + 1 \) for \( 0 < t < \infty \).

(a) Find an equation in \( x \) and \( y \) that describes \( C \).

Solution: Since \( x^2 = \frac{1}{t} \) we get \( y = x^2 + 1 \).

(b) Graph \( C \), indicating the direction of motion of an object whose location is given by the above parametric equations.

Solution:

(c) Is \( (0, 1) \) a point on \( C \)? Why?

Solution: No, because \( \frac{1}{\sqrt{t}} \) is never equal to 0.
11. The location of an object at time $t$ is given by $x(t) = e^t + e^{-t}$ and $y(t) = 5 - 2t$. Find the distance traveled by the object from $t = 0$ to $t = 1$.

Solution: The distance traveled is the length of the curve.

$$x' = e^t - e^{-t} \quad \Rightarrow \quad (x')^2 = e^{2t} - 2 + e^{-2t}$$

$$y' = -2 \quad \Rightarrow \quad (y')^2 = 4$$

$$\left((x')^2 + (y')^2\right) = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$$

$$\text{distance} = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

$$= \int_0^1 \sqrt{(e^t + e^{-t})^2} \, dt$$

$$= \int_0^1 (e^t + e^{-t}) \, dt$$

$$= (e^t - e^{-t}) \bigg|_0^1$$

$$= (e - e^{-1}) - (1 - 1)$$

$$= e - e^{-1}$$

12. Find the surface area of a sphere of radius $a$ by using the standard parametrization for the upper half of a circle of radius $a$ and rotating that about the $x$-axis.

Solution: The usual parametrization of the circle of radius $a$ is $x(t) = a \cos t$ and $y(t) = a \sin t$. Taking $0 \leq t \leq \pi$ gives us the upper half of the circle. This is the curve $C$ that we want to rotate about the $x$-axis.

$$x'(t) = -a \sin t \quad \Rightarrow \quad (x'(t))^2 = a^2 \sin^2 t$$

$$y'(t) = a \cos t \quad \Rightarrow \quad (y'(t))^2 = a^2 \cos^2 t$$

$$\left((x'(t))^2 + (y'(t))^2\right) = a^2 \sin^2 t + a^2 \cos^2 t = a^2$$

$$\text{surface area} = 2\pi \int_0^\pi y(t) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

$$= 2\pi \int_0^\pi a \sin t \sqrt{a^2} \, dt$$

$$= 2a^2 \pi \int_0^\pi \sin t \, dt$$

$$= 2a^2 \pi \left[-\cos t\right]_0^\pi$$

$$= 2a^2 \pi (1 + 1)$$

$$= 4a^2 \pi$$