Solve the differential equations in problems 1–5. It is not necessary to solve completely for $y$, but you should eliminate the logarithms from your answers. If an initial condition is given, find, if possible, the unique solution which satisfies the condition. If no such unique solution exists, tell whether there is no solution or an infinite number of them.

1. $y' - 6y \tan 2x = -3y^{2/3}$

   Solution. This equation is not separable, homogeneous, exact, or linear. It is Bernoulli. First put the equation into the proper form to determine the substitution by multiplying by $y^{-2/3}$.

   $$y^{-2/3}y' - 6y^{1/3} \tan 2x = -3$$

   Let $w = y^{1/3}$. Then $w' = (1/3)y^{-2/3}y'$ or $3w' = y^{-2/3}y'$.

   $$3w' - 6w \tan 2x = -3$$
   $$w' - 2w \tan 2x = -1$$

   Now the equation is linear and we can set $P(x) = -2 \tan 2x = -2 \sin 2x / \cos 2x$. Then $\int P(x) \, dx = \ln(\cos 2x)$ so $\mu = \cos 2x$.

   $$w' \cos 2x - 2w \sin 2x = -\cos 2x$$
   $$(w \cos 2x)' = -\cos 2x$$

   $$w \cos 2x = -\int \cos 2x \, dx = -\frac{1}{2} \sin 2x + c$$

   $$w = c \sec 2x - \frac{1}{2} \tan 2x$$

   $$y^{1/3} = c \sec 2x - \frac{1}{2} \tan 2x$$
2. \((2xy + 3y^2) \, dx - (2xy + x^2) \, dy = 0, \quad y(0) = 0\)

**Solution.** This equation is not separable, exact, linear, or Bernoulli. It is homogeneous. The coefficients are both homogeneous of order 2. So set \(y = ux\) and \(dy = u \, dx + x \, du\).

\[
(2ux^2 + 3u^2x^2) \, dx - (2ux^2 + x^2) \, (u \, dx + x \, du) = 0
\]

\[
x^2(2u + 3u^2) \, dx - x^2(2u + 1)(u \, dx + x \, du) = 0
\]

\[
(2u + 3u^2) \, dx - (2u + 1)(u \, dx + x \, du) = 0
\]

\[
(2u + 3u^2 - 2u^2 - u) \, dx - (2u + 1)x \, du = 0
\]

\[
(u^2 + u) \, dx = (2u + 1)x \, du
\]

\[
\frac{2u + 1}{u^2 + u} \, du = \frac{1}{x} \, dx
\]

\[
\int \frac{2u + 1}{u^2 + u} \, du = \int \frac{1}{x} \, dx
\]

\[
\ln |u^2 + u| = \ln |x| + c
\]

\[
u^2 + u = cx
\]

\[
\left(\frac{y}{x}\right)^2 + \frac{y}{x} = cx
\]

\[
y^2 + xy = cx^3
\]

Now set \(x = y = 0\). We get \(0 + 0 = 0\), which is correct, but we cannot solve for \(c\). In fact, \(y^2 + xy = cx^3\) gives us a solution for the given initial value problem for all values of \(c\), so there are an infinite number of solutions.
3. \( x(x - 1) \, dy - y \, dx = 0, \quad y(2) = 1/2 \)

Solution. This equation is separable. It is not homogeneous, exact, or Bernoulli. It is also linear, but we will use the technique of separation of variables.

\[
\frac{1}{y} \, dy = \frac{1}{x(x - 1)} \, dx
\]

\[
\int \frac{1}{y} \, dy = \int \frac{1}{x(x - 1)} \, dx
\]

We must use the method of partial fraction expansion to evaluate the integral with respect to \( x \). If \( \frac{1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1} \) then \( A = \frac{1}{x - 1} \bigg|_{x=0} = -1 \) and \( B = \frac{1}{x} \bigg|_{x=1} = 1 \).

\[
\int \frac{1}{y} \, dy = \int \left( \frac{1}{x - 1} - \frac{1}{x} \right) \, dx
\]

\[
\ln |y| = \ln |x - 1| - \ln |x| + c
\]

\[
y = c \left( \frac{x - 1}{x} \right)
\]

Now set \( x = 2 \) and \( y = 1/2 \). We get \( 1/2 = c(1/2) \), so \( c = 1 \) and the unique solution to the initial value problem is \( y = \frac{x - 1}{x} \).

4. \( y' - ay = be^{ax} \), where \( a \) and \( b \) are constants.

Solution. This equation is not separable, homogeneous, or exact. It is linear. Set \( P(x) = -a \). Then

\[
\int P(x) \, dx = -ax \quad \text{and} \quad \mu = e^{-ax}.
\]

\[
e^{-ax} y' - ae^{-ax} y = b
\]

\[
(e^{-ax} y)' = b
\]

\[
e^{-ax} y = \int b \, dx
\]

\[
e^{-ax} y = bx + c
\]

\[
y = bxe^{ax} + ce^{ax}
\]
5. \((xy^2 + y - x) \, dx + (x^2y + x - y^2) \, dy = 0\)

Solution. This equation is not separable, homogeneous, linear, or Bernoulli. We will check to make sure that it is exact. Let \(M = xy^2 + y - x\) and \(N = x^2y + x - y^2\). Then \(\frac{\partial M}{\partial y} = 2xy + 1\) and \(\frac{\partial N}{\partial x} = 2xy + 1\). Therefore the equation is exact.

\[
f(x, y) = \int (xy^2 + y - x) \, dx
\]
\[
= \frac{1}{2}x^2y^2 + xy - \frac{1}{2}x^2 + g(y)
\]

But we want the partial of \(f\) with respect to \(y\) to be \(N\).

\[
x^2y^2 + x + g'(y) = x^2y + x - y^2
\]
\[
g'(y) = -y^2
\]
\[
g(y) = -\int y^2 \, dy = -\frac{1}{3}y^3
\]

The solution is \(\frac{1}{2}x^2y^2 + xy - \frac{1}{2}x^2 - \frac{1}{3}y^3 = c\).
6. Show that if \( f \) is a differentiable function and \( f(0) \neq 0 \) then \( f \) and \( e^x f \) are linearly independent on \( \mathbb{R} \).

Solution. We will use the Wronskian for this.

\[
W(f, e^x f) = \begin{vmatrix} f & e^x f \\ f' & e^x f' \end{vmatrix} = e^x f^2 + e^x f f' - e^x f f' = e^x f^2
\]

Therefore the Wronskian is nonzero when \( x = 0 \) and the functions \( f \) and \( e^x f \) are linearly independent.

7. Show that the functions \( f(x) = x^2 - x, g(x) = x^2 + x, \) and \( h(x) = x^2 - 5x \) are linearly dependent. Do not use the Wronskian.

Solution. We must find numbers \( a, b, \) and \( c, \) not all of which are 0, so that \( af(x) + bg(x) + ch(x) = 0 \) for all \( x \). Now \( af(x) + bg(x) + ch(x) = a(x^2 - x) + b(x^2 + x) + c(x^2 - 5x) = (a + b + c)x^2 + (-a + b - 5c)x \). If this equation equals 0 for all \( x \) then \( a + b + c = 0 \) and \( -a + b - 5c = 0 \). Since we have two equations and three unknowns, we will guess a value for one of the unknowns. Let \( c = 1 \). Then \( a + b = -1 \) and \( -a + b = 5 \).

Adding these two equations gives us \( 2b = 4 \), so \( b = 2 \) and \( a = -3 \). Since \( -3f(x) + 2g(x) + h(x) = 0 \) for all \( x \) we know that \( f, g, \) and \( h \) are linearly dependent.