1. Define the following terms.

(a) \( m[f;I] \)

Solution. If \( I \) is an interval in \( \mathbb{R} \) and \( f \) is a bounded real-valued function defined on \( I \) then
\[
m[f;I] = \inf\{f(x) : x \in I\}.
\]

(b) subdivision of an interval

Solution. A subdivision of an interval \([a,b]\) is a finite set \( \sigma = \{x_0, x_1, \ldots, x_n\} \) where
\[a = x_0 < x_1 < \cdots < x_n = b.\]

(c) \( \int_a^b f \)

Solution. If \( a, b \in \mathbb{R} \) with \( a < b \) and \( f \) is a bounded real-valued function defined on \([a,b]\) then
\[
\int_a^b f \, dx = \inf_{\sigma} M[f;\sigma].
\]

(d) uniformly convergent sequence of functions

Solution. Let \( X \) be a set and let \( (f_n) \) be a sequence of real-valued functions defined on \( X \). Let \( f : X \to \mathbb{R} \). Then \( (f_n) \) converges uniformly to \( f \) if and only if for every \( \epsilon > 0 \) there is \( N \in \omega \) such that \( |f(a) - f_n(a)| < \epsilon \) for all \( n \geq N \) and all \( a \in X \).
In the following problems let \( a, b \in \mathbb{R} \) with \( a < b \) and let \( f, g : [a, b] \to \mathbb{R} \).

2. Prove that if \( f \in \mathbb{R}[[a, b]] \) and \( f(x) \) is never 0 on \([a, b] \) then \( 1/f \in \mathbb{R}[[a, b]] \).

Solution. Let \( A \) be the set of points in \([a, b] \) at which \( f \) is discontinuous and let \( B \) be the set of points in \([a, b] \) at which \( 1/f \) is discontinuous. Then \( A \) has measure 0 because \( f \in \mathbb{R}[[a, b]] \). If \( f \) is continuous at some \( x \in [a, b] \) then \( 1/f \) is also continuous at \( x \) because \( f(x) \neq 0 \). Thus, if \( x \in B \) then \( x \in A \), or \( B \subseteq A \). It follows that \( B \) has measure 0 and that \( 1/f \in \mathbb{R}[[a, b]] \).

3. Prove that if \( f \) is continuous on \([a, b] \) and \( \int_a^b |f| \, dx = 0 \) then \( f(x) = 0 \) for all \( x \in [a, b] \).

Solution. If \( f \) is continuous on \([a, b] \) then so is \(|f| \). We proved in homework that if a function \( g \) is integrable and nonnegative on \([a, b] \) and there is \( x \in [a, b] \) such that \( g(x) > 0 \) then \( \int_a^b g \, dx > 0 \). But \(|f| \) is integrable and nonnegative on \([a, b] \) and \( \int_a^b |f| \, dx = 0 \), so \(|f(x)| \) must be 0 for all \( x \in [a, b] \). Thus \( f(x) = 0 \) for all \( x \in [a, b] \).
4. Let \( A \) be a nonempty bounded subset of \( \mathbb{R} \) and let \( c \in \mathbb{R} \) with \( c > 0 \). Set \( cA = \{ cx : x \in A \} \). Prove that \( \sup cA = c \sup A \).

Solution. If \( x \in A \) then \( x \leq \sup A \). Thus \( cx \leq c \sup A \). It follows that \( \sup cA \leq c \sup A \). If \( x \in A \) then \( cx \leq \sup cA \). Thus \( x \leq (1/c) \sup cA \). It follows that \( \sup A \leq (1/c) \sup cA \) or \( c \sup A \leq \sup cA \). Therefore \( \sup cA = c \sup A \).

5. Prove that if \( f \in \mathcal{R}[a, b] \) and \( c \in \mathbb{R} \) with \( c > 0 \) then \( \int_a^b cf \, dx = c \int_a^b f \, dx \). You may assume that \( cf \in \mathcal{R}[a, b] \).

Solution.

\[
\int_a^b cf \, dx = \sup_{\sigma} \sum_{k=1}^{n} m(cf; I_k)^{|I_k|}
\]
\[
= \sup_{\sigma} \sum_{k=1}^{n} \left( \inf_{x \in I_k} cf(x) \right)^{|I_k|}
\]
\[
= \sup_{\sigma} \sum_{k=1}^{n} \left( c \inf_{x \in I_k} f(x) \right)^{|I_k|}
\]
\[
= \sup_{\sigma} c \sum_{k=1}^{n} m[f; I_k]|I_k|
\]
\[
= c \sup_{\sigma} \sum_{k=1}^{n} m[f; I_k]|I_k|
\]
\[
= c \int_a^b f \, dx
\]
6. Let $a \in \mathbb{R}$ and let $\langle a_n \rangle$ be a sequence in $\mathbb{R}$ that converges to $a$. Prove that \{a_n : n \in \omega\} is a set of measure 0.

**Solution.** Let $\epsilon > 0$ and set $J = (a - \epsilon/4, a + \epsilon/4)$. Since $\lim_{n \to \infty} a_n = a$ there is $N \in \omega$ such that $a_n \in J$ for all $n \geq N$. For $k = 0, \ldots, N - 1$ set $I_k = (a_k - \epsilon/(4N), a_k + \epsilon/(4N))$. Set $I_N = J$. Then $\{a_n : n \in \omega\} \subseteq \bigcup_{k=0}^{N} I_k$ and $\sum_{k=0}^{N} |I_k| = \epsilon$.

7. Let $E$ be a set and for every $n \in \omega$ let $f_n : E \to \mathbb{R}$. Let $f : E \to \mathbb{R}$ and let $c \in \mathbb{R}$. Prove that if $\langle f_n \rangle$ converges uniformly to $f$ then $\langle cf_n \rangle$ converges uniformly to $cf$.

**Solution.** Let $\epsilon > 0$. There is $N \in \omega$ such that $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$ and all $x \in E$. If $c = 0$ then $cf_n(x) = 0$ for all $n \in \omega$ and all $x \in E$ and $cf(x) = 0$ for all $x \in E$. Obviously then $|cf(x) - cf_n(x)| < \epsilon$ for all $n \geq N$ and all $x \in E$. Assume that $c \neq 0$. Let $M \in \omega$ such that $|f(x) - f_n(x)| < \epsilon |c|$ for all $n \geq M$ and all $x \in E$. If $n \geq M$ and $x \in E$ then $|cf(x) - cf_n(x)| = |c||f(x) - f_n(x)| < |c|\epsilon/|c| = \epsilon$. So $\langle cf_n \rangle$ converges uniformly to $cf$. 
8. Let \( f_n(x) = e^{-nx} \) for all \( n \in \omega \). Does \( \langle f_n \rangle \) converge uniformly on \([0, 1]\)? Why?

Solution. If \( 0 < x \) then \( \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} e^{-nx} = 0 \). But \( \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} 1 = 1 \). Thus each \( f_n \) is continuous, but \( \lim_{n \to \infty} f_n \) is not continuous and the convergence cannot be uniform.

9. For every \( n \in \omega \) and every \( x \in [0, 1] \) set \( f_n(x) = \begin{cases} 2^n \sin(2^n \pi x) & 0 \leq x < 2^{-n} \\ 0 & 2^{-n} \leq x \leq 1 \end{cases} \). Set \( f(x) = 0 \) for all \( x \in [0, 1] \). Then \( \langle f_n \rangle \) converges to \( f \). Is the convergence uniform? Why?

Solution. Each \( f_n \) is continuous, and therefore integrable. But \( \lim_{n \to \infty} f_n = f \) is also continuous and integrable, so that tells us nothing. Let \( n \in \omega \).

\[
\int_0^1 f_n \, dx = \int_0^{2^{-n}} 2^n \sin(2^n \pi x) \, dx + \int_{2^{-n}}^1 0 \, dx \\
= \left[ -\frac{1}{\pi} \cos(2^n \pi x) \right]_0^{2^{-n}} + 0 \\
= \frac{2}{\pi}
\]

But \( \int_0^1 f \, dx = \int_0^1 0 \, dx = 0 \). So \( \lim_{n \to \infty} \int_0^1 f_n \, dx = \frac{2}{\pi} \neq 0 = \int_0^1 (\lim_{n \to \infty} f_n) \, dx \) and the convergence is not uniform.