**Calendar Problems**

**Easy: (25 points)**

The University of Dayton Math Club would like to hold another Math Competition in 214 days from today March 16, 2013. Will this day work for our *Saturday* event?

**Medium: (50 points)**

Given a calendar of March 2013, figure out which set of 5 dates forming a cross have the sum of 90? For example $1 + 7 + 8 + 9 + 15 = 40$

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**Hard: (100 points)**

If a baby was born September 20, 2005 how many days old is the child? Count his birth date as zero and include today’s date.
Calendar Answers

Easy Answer:

NO, 214 (mod 7) = 4 meaning 214 days from today is a Wednesday

Medium Answer:

\[ n + (n+6) + (n + 7) + (n + 8) + (n + 14) = 90 \]
\[ 5n + 35 = 90 \]
\[ 5n = 55 \]
\[ n = 11 \]

So the dates: 11\(^{th}\), 17\(^{th}\), 18\(^{th}\), 19\(^{th}\), and 25\(^{th}\)

Hard Answer:

2005 - 102 days
2008, and 2012 - 366 days (leap years)
2013 – 75 days
Grand Total: 2,734
Counting Problems

Easy: (25 points)

A University of Dayton sophomore needs to take one course of physics, one course of mathematics, one course of history, and one course of psychology. There are 5 physics classes offered, 7 mathematics courses, 9 type of history classes, and 6 psychology options. How many different schedules could this student have? That is, in how many different ways can the four courses be selected?

Medium: (50 points)

At a certain company every employer has a 6 digit ID number. The first three digits denote one of the three departments of the company (111), (222), or (333). The last 3 digits can be any number; however the ID cannot end with a 0. Are there enough ID numbers for a company with 2,000 employees? How many ID numbers are there?

Hard: (100 points)

Assume there are seven parking spaces next to each other in a parking lot. Seven cars need to be parked by the attendant. Three of the cars are expensive sport cars. Assuming that the attendant parks the cars at random, in how many ways can they be parked so that the three expensive sport cars are parked adjacent to one another?
Counting Problems

Easy Answer: $5 \times 7 \times 9 \times 6 = 1890$ possible schedules for the UD student.

Medium Answer: YES; $3(10)(10)(9) = 2,700$ ID numbers.

Hard Answer: $5(4!)(3!) = 360$
Geometry Problems

Easy: (25 points)

Given BC = $\sqrt{3}$, AB = $2\sqrt{3}$. Find AD, DB, CD, and AC.

Medium: (50 points)

Three logs of diameter 8 cm are stacked in a pile in the following manner. How tall is the pile?

Hard: (100 points)

Two adjacent sides of the square with side length of 1 and two sides of an equilateral triangle bisect each other. Find the area of the equilateral triangle.
Geometry Answers

Easy Answer: Given $BC = \sqrt{3}$, $AB = 2\sqrt{3}$. Find $AD$, $DB$, $CD$, and $AC$.

AC can be found using the Pythagorean Theorem and the remaining segments can be found using properties of 30-60-90 triangles.

\[ AC = 3, \ DB = \frac{\sqrt{3}}{2}, \ CD = \frac{3}{2}, \ \text{and} \ AD = \frac{3\sqrt{2}}{2} \]

Medium Answer: Three logs of diameter 8 cm are stacked in a pile in the following manner. How tall is the pile?

By creating an equilateral triangle from the centers of the three circles, the height of the triangle can be found to be $4\sqrt{3}$. Then, the pile of logs will be $8 + 4\sqrt{3}$ cm.

Hard Answer: Two adjacent sides of the square with side length of 1 and two sides of an equilateral triangle bisect each other. Find the area of the equilateral triangle.

Start by constructing a segment joining the two intersection points. This segment has length of $\frac{\sqrt{2}}{2}$ by the Pythagorean Theorem. Thus, the sides of the triangle are $\sqrt{2}$. Then, the height is $\frac{\sqrt{6}}{2}$. Thus, the area is $\frac{\sqrt{3}}{2}$. 

Algebra Problems

Easy: (25 points)

Does a > b imply that 1/a < 1/b for all nonzero real numbers? (Yes/ No and explain)

Medium: (50 points)

Three cowboys entered a saloon. The first ordered 4 sandwiches, a cup of coffee, and 10 donuts for $8.45. The second ordered 3 sandwiches, a cup of coffee, and 7 doughnuts for $6.30. How much did the third cowboy pay for a sandwich, a cup of coffee, and a doughnut?

Hard: (100 points)

Given that a + b + c = 5 and ab + bc + ac = 5, what is the value of a^2 + b^2 + c^2?
**Algebra Answers**

**Easy Answer:** Does $a > b$ imply that $\frac{1}{a} < \frac{1}{b}$ for all nonzero real numbers? (Yes/ No and explain)

No. Counterexample: If $a = 1$ and $b = -1$, then $a > b$ and $\frac{1}{a} > \frac{1}{b}$.

**Medium Answer:** Three cowboys entered a saloon. The first ordered 4 sandwiches, a cup of coffee, and 10 donuts for $8.45$. The second ordered 3 sandwiches, a cup of coffee, and 7 doughnuts for $6.30$. How much did the third cowboy pay for a sandwich, a cup of coffee, and a doughnut?

\[
2(4s + 1c + 10d = 8.45) \rightarrow 8s + 2c + 20d = 16.90 \\
3(3s + 1c + 7d = 6.30) \rightarrow 9s + 3c + 21d = 18.90 \\
\text{Then subtracting the first equation from the second yields: } 1s + 1c + 1d = 2.00. \\
\text{So, }$2.00. \\

**Hard Answer:** Given that $a + b + c = 5$ and $ab + bc + ac = 5$, what is the value of $a^2 + b^2 + c^2$?

\[
(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc. \\
\text{So, } a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+ac+bc) \\
\text{So, } a^2+b^2+c^2 = 5^2 - 2(5) = 15.\]
History of Math Problems

Easy: (25 points)

In Base 4 system, find the sum of:

\[(112233)_4 + (332211)_4\]

Medium: (50 points)

\[\frac{5}{8}\] as a decimal fraction (i.e. using base 10) would be \[\frac{6}{10} + \frac{2}{100} + \frac{5}{1000}\].

The Babylonians used base 60. Write \[\frac{5}{8}\] as a base 60 fraction.

Hard: (100 points)

The Fibonacci sequence is as follows: 1, 1, 2, 3, 5, 8, 13, 21, 34, …

It is recursively defined as \(F_n = F_{n-1} + F_{n-2}\) where \(F_0 = 1\) and \(F_1 = 1\).

Demonstrate how to construct the Fibonacci sequence from Pascal’s Triangle.

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]
History of Math Answers

Easy Answer: \((1111110)_4\)

Medium Answer: \(\frac{5}{8} = \frac{37}{60} + \frac{30}{60^2}\)

Hard Answer: 6 15 20 15 6 1
Logic Problems

Easy: (25 points)

There are five people waiting in line for free T-shirts in Kennedy Union Plaza. Their names are Steve, Christopher, Carly, Mallory, and Emily. The person with the longest name is standing at the front of the line. Carly is standing behind Christopher and in front of Emily. Mallory is after Carly but before Steve. List the names of the people in the order they are in the line.

Medium: (50 points)

You are traveling with a horse, a cow, and a goat. From prior experience, you know that if you turn your back on these animals, the horse will chase the goat, and the cow will chase the horse. This is not a problem if you do not leave these animals alone with each other. You come upon a river that is too deep for even the cow or horse to wade across, but you find a boat that can hold you and one animal, but only one. Step by step, how do you get the animals all to the other side, without any animals chasing the others away?

Hard: (100 points)

How many ways can you fill the middle triangle with three of the numbers 1 through 9, using each one exactly once, so that the outer numbers add up to twice the inner numbers?
Logic Problems

Easy Answer:

Medium Answer:

Step 1: take the horse, because the goat and cow get along.
Step 2: go back and take the goat.
Step 3: leave the goat and take the horse back.
Step 4: drop off the horse and pick up the cow.
Step 5: leave the cow and go back for the horse.
Step 6: pick up the horse.

Everyone is across.

Hard Answer:

8

1 5 9
1 6 8
2 4 9
2 5 8
2 6 7
3 4 8
3 5 7
4 5 6
Number Theory Problems

Easy: (25 points)

What is the greatest common factor of the pair of numbers 84 and 147?

i.e. GCF(84,147)

Medium: (50 points)

Find all positive integers $n$ for which $|3n - 4|$, $|4n - 5|$, and $|5n - 3|$ are all prime numbers.

Hard: (100 points)

Find all primes $p$ and $q$ such that $x^2 - px + q = 0$ has two distinct positive integer roots.
Number Theory Answers

Easy Answer: 21

Medium Answer: $n = 1, n = 2$

Hard Answer: $p = 3, q = 2$
Trigonometry Problems

Easy: (25 points)

Find \( \sin(\theta) \), \( \cos(\theta) \), and \( \tan(\theta) \) for the following triangle.

![Triangle with sides 26 and 10 and angle \( \theta \)]

Medium: (50 points)

Find \( y \) in the following triangle.

![Complex triangle with angles 45°, 30°, and 60° and side 20]

Hard: (100 points)

Solve \( 4\sec^2(x) - \tan^4(x) = 7 \) on the interval \((-\pi/2, \pi/2)\).
Trigonometry Answers

Easy Answer:

\[26^2 = 10^2 + b^2\]

\[b = 24\]

\[\sin(\theta) = \frac{O}{H} = \frac{24}{26}\]

\[\cos(\theta) = \frac{A}{H} = \frac{10}{26}\]

\[\tan(\theta) = \frac{O}{A} = \frac{24}{10}\]

Medium Answer:

\[x = 20\sin(30^\circ) = 10\]

\[a = \frac{10}{\tan(45^\circ)} = 10\]

\[y = \frac{10}{\cos(30^\circ)} = \frac{20}{3}\sqrt{3}\]

Hard Answer:

Solve \(4\sec^2(x) – \tan^4(x) = 7\) on the interval \((-\pi/2, \pi/2)\)

Use \(1 + \tan^2(x) = \sec^2(x)\)

\[4(1+\tan^2(x)) – \tan^4(x) = 7\]

\[4 + 4\tan^2(x) – \tan^4(x) = 7\]

\[\tan^4(x) – 4\tan^2(x) – 4 = -7\]

\[\tan^4(x) – 4\tan^2(x) + 3 = 0\]

\[(\tan^2(x)-1) (\tan^2(x)-3) = 0\]

\[\tan^2(x)-1 = 0 \text{ or } \tan^2(x) – 3 = 0\]

\[\tan(x) = \sqrt{1} \text{ or } \tan(x) = \sqrt{3}\]

\[x = \pm\pi/4 \text{ or } x = \pm\pi/3\]
Graph Theory Problems

Easy: (25 points)
An example of a minimal spanning tree is the cheapest way to build roads between cities. If the circles with the letters A-F represent the cities and the lines represent roads, and the numbers indicate the cost of the road, find the minimal spanning tree, or cheapest way to connect all 6 cities, for the following figure. (each city must touch at least one road)

Medium: (50 points)
A Eulerian path is a trail in a graph which visits every edge exactly once. Is there a eulerian path for the following figure?

Hard: (100 points)
A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. How many edges are there in a complete graph with 200 nodes?