

Geometry

Problem 1: (100 points)

Given:

HG parallel to AC, line segment BD is an angle bisector of triangle ABC; $AG=AD$; $HC=CD$.

Find HG.

Solution:

We know $AG/12=HC/9$, so $12(HC)=9(AG)$. We know $HC=CD$ and $AG=AD$. So we can substitute into our equation, $12(CD)=9(AD)$. We know that $CD=(7-(AD))$. So we can substitute that into our equation. $12(7-AD)=9(AD)$. Which leads to $84-12AD=9AD$. $84=21AD$. So $4=AD$. Since $AD=AG$ we can now write $8/12=HG/7$. So $56=12HG$. And $HG=14/3$.

Problem 2: (50 points)

Find the area of a regular hexagon if the area of an equilateral triangle with the same perimeter is $3\sqrt{3}$.

Solution:

We know from theorems that the area of a hexagon is $1/2$ (apothem times perimeter), and the area of an equilateral triangle is either $1/2$ (base times height) or the $(\text{square root of } 3 \text{ times the side}^2)/4$. We know that the area of the equilateral triangle is $3(\text{square root of } 3)$, so we can set this equal to the equation of the equilateral triangle. By working out this equation we get $\text{side}^2=3 \text{ times } 4$. This can be simplified to $\text{side} = \text{square root of } 12$ (or $\text{side} = 2 \text{ times the square root of } 3$). Now that we know one side of the equilateral triangle, we can multiply that by three to obtain the perimeter, which is $6 \text{ times the square root of } 3$, which is also the same perimeter as the hexagon. Since there are six sides to a hexagon we know that the sides of the hexagon must be square root of 3, if the perimeter is $6(\text{square root of } 3)$. In a regular hexagon there are 6 equilateral triangles, so one side is square root of 3. We already stated the formula to find the area of an equilateral triangle, now we have $(3^{1/2} \text{ times } (3^{1/2})^2)/4$ as the area of one equilateral triangle inside the hexagon. If we multiply this by 6 (the number of equilateral triangles in the hexagon) we obtain the final solution $(9 \text{ times } 3^{1/2})/4$.

Problem 3: (25 points)

Find the area of the shaded region in the following figure. The dots represent the centers of the circles.

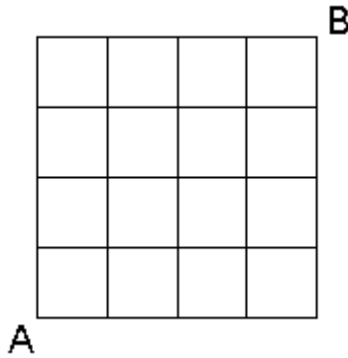
Solution:

The area of the big circle (circle Q) is 36π . Half of the area of circle Q is 18π . The smallest circle (circle R) has the area 4π , and half of that is 2π . The area of the middle-sized circle (circle P) is 16π , and half of that is 8π . Take half the area of circle Q (18π) and subtract from it half the area of circle P (8π), and add half the area of circle R (2π) to the total to obtain the answer 12π .

Algebra

Problem 1: (100 points)

A policeman has to patrol a certain number of streets, arranged in a 4 by 4 block.



He tends to walk from one corner of this block (A) to the opposite corner (B), but tries to take a different route each time. In this way, he hopes to deter would-be criminals and also gives himself something intellectual to do, to pass the oh-so-long night hours...

He does like to take an 'efficient' route, as short as possible, but still finds he has lots of choice...

How many such routes are there from A to B ?

Solution:

There are 70 different routes altogether.

An 'efficient' route always travels in an 'easterly' or 'northern' direction.

There are therefore 8 street lengths to travel, of which 4 are 'N's and 4 'E's.

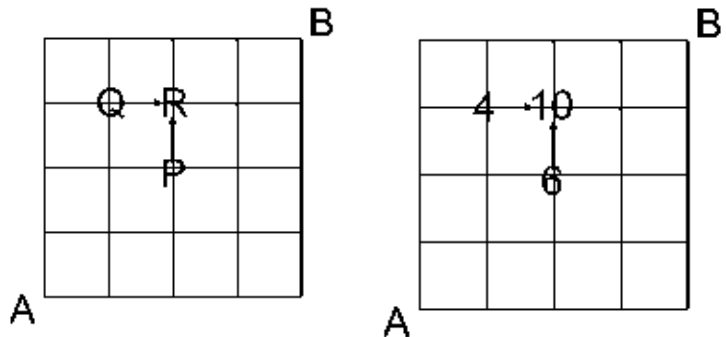
The problem is therefore to find how many ways there are to place 4 'N's within a sequence of 8 'slots'.

There are, of course, ${}^8C_4 = 70$ ways to do this.

One approach to the problem is to begin from A and to label the adjacent street junctions with the number of ways to get that far from A.

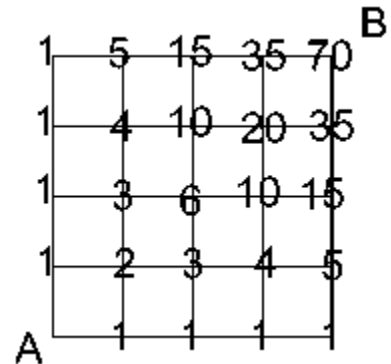
Students may well notice soon that each label is the sum of the labels to the left and below it.

For example, the only way to get to R is from P and Q, so the total number of ways to get to R is the sum of the number of ways that got him as far as P or Q.



Once you realize that each of the exterior junctions will be labeled with a '1', the rest follow, in a Pascal's triangle array, albeit on the diagonal...

The symmetry is pleasing.

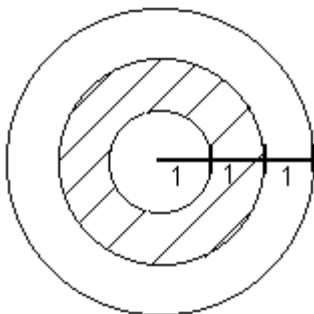


The street junctions along the 'leading' diagonal, from A to B, have labels of the form

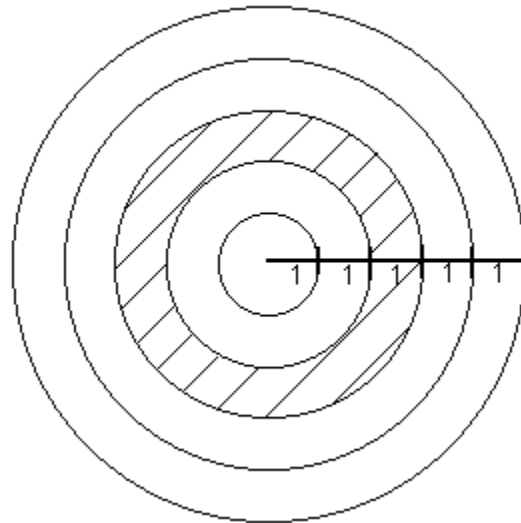
${}^{2n}C_n$, where n is the number of blocks traveled east (or north).

Problem 2: (50 points)

What fraction of each target area are the shaded rings?



Target radius 3



Target radius 5

Solution:

$$A \quad \frac{\pi(2^2 - 1^2)}{\pi(3^2)} = \frac{3\pi}{9\pi} = \frac{1}{3}$$

$$B \quad \frac{\pi(3^2 - 2^2)}{\pi(5^2)} = \frac{5\pi}{25\pi} = \frac{1}{5}$$

Problem 3: (25 points)

There are these three numbers...

When added together, in pairs only, the sums come to 11, 17 and 22.

What are the three numbers?

Solution:

$$(a+b) + (b+c) + (c+a) = 11+17+22 = 50$$

$$\text{So, } 2(a+b+c) = 50$$

$$\text{And, } a+b+c = 25$$

$$\text{Now, since } a+b = 11, \text{ so } \mathbf{c = 14}$$

$$\text{Similarly, } b+c = 17, \text{ so } \mathbf{a = 8}$$

$$\text{And } c+a = 22, \text{ so } \mathbf{b = 3}$$

Graph Theory

Problem 1: (100 points)

Suppose you and your husband attend a party with 3 other married couples. Several handshakes took place. No one shook hands with himself (or herself) or with his (or her) spouse, and no one shook hands with the same person more than once. After all the handshaking was completed, suppose you asked each person, including your husband, but not including yourself, how many hands he or she had shaken. Each person gave a different answer.

How many hands did you shake? How many hands did your husband shake?

(Hint: draw a graph with as many vertices as there are people, and draw an edge from one vertex to another to indicate a handshake. When you are done, the number of edges attached to each vertex is the number of handshakes that person participated in.)

Solution:

You shook 3 hands, and your husband shook 3 hands

Problem 2: (50 points)

Below is a picture of the city of Königsberg, Germany. The city consists of 4 landmasses (A, B, C, and D), and there are bridges connecting the landmasses. The people of Königsberg want tourists to see their lovely city, so they challenge visitors to a walking tour. The challenge is to start at any one-land mass, and visit all other landmasses by crossing every bridge once and only once. If this is possible, which land mass must you start at?

Solution:

It's not possible. There are more than 2 vertices (land masses) with odd degree (an odd number of bridges connecting them).

Problem 3: (25 points)

A coloring of a graph is an assignment of 'colors' to the vertices of the graph so that no 2 adjacent vertices (i.e. vertices connected by an edge) are the same color. In the graph below, determine the smallest number of colors needed to color it (by the way, this is called the chromatic number).

Solution:

4

History of Math

Problem 1: (100 points)

In the past different civilizations used different number systems. The Ancient Babylonians used the (base 60) number system. Today we use the (base 10) number system. Multiply $3212_4 * 322_4$ (keep your answer in base 4).

Solution:

Convert 3212 and 322 to base 10. This gives you $230 * 58$. The product of those two is 13340. Then convert 13340 back to base 4 which yields 3100130_4 .

Problem 2: (50 points)

The ancient Egyptians had a unique way of writing fractional numbers. They would write their fractions as the sum of several distinct unit fractions (ex: $5/6 = 1/2 + 1/3$). Write $15/32$ as a sum of distinct unit fractions.

Solution:

Break the fraction up so that $15/32 = 1/32 + 14/32$ and continue this process until you get: $1/32 + 1/16 + 1/8 + 1/4$, or $1/3 + 1/8 + 1/96$

Problem 3: (25 points)

The golden ratio is a very special number in mathematics. The tendency for the golden ratio to appear in nature is explained by the fact the golden ratio is an aesthetically pleasing one. What is mathematically special about the golden ratio is that it is equal to its own reciprocal plus one ($\phi = 1/\phi + 1$ where ϕ "phi" is the golden ratio). What is the positive numerical value of ϕ ?

Solution:

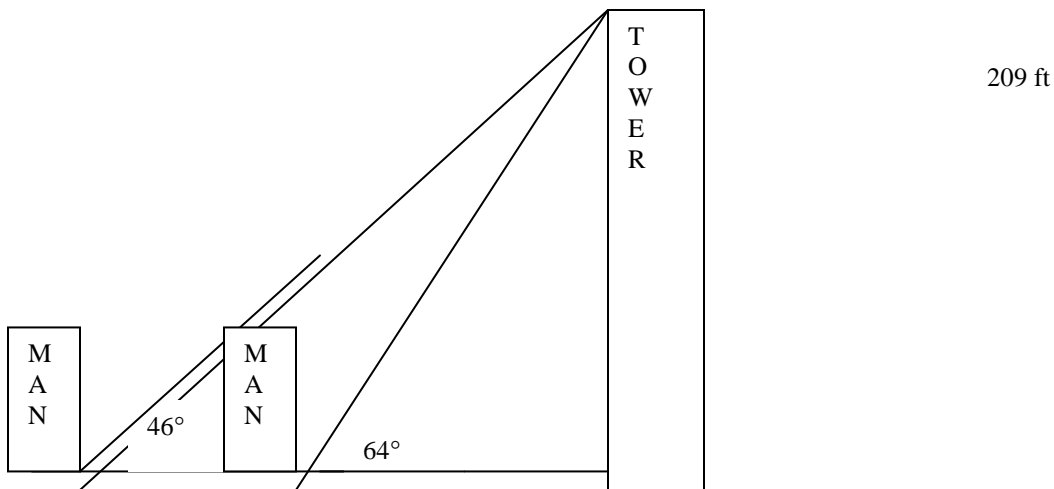
Multiply both sides by the golden ration which gives you $\phi^2 = 1 + \phi$ OR $\phi^2 - \phi - 1 = 0$. Then using the quadratic equation you get that $\phi = (1 + \sqrt{5})/2$.

Trigonometry

Problem 1: (100 points)

A man standing near a radio station antenna observes that the angle of elevation to the top of the antenna is 64° . He then walks 100 ft further away and observes that the angle of elevation to the top of the antenna is 46° . Find the height of the antenna to the nearest foot.

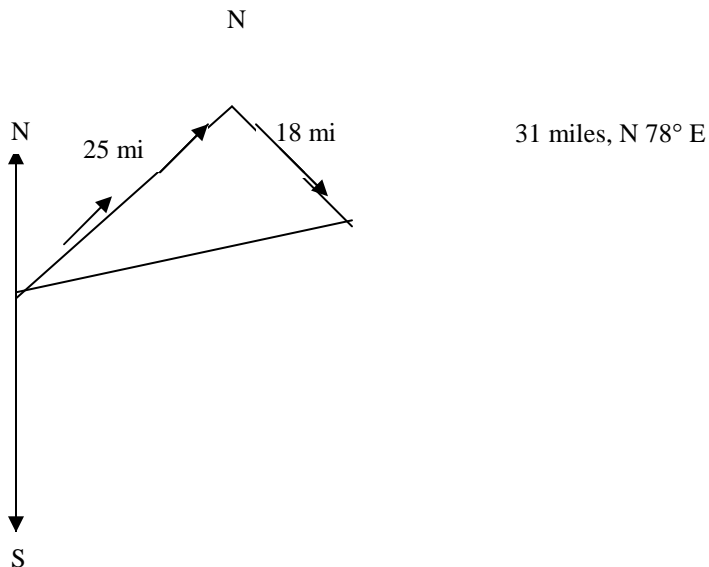
Solution:



Problem 2: (50 points)

A boat leaves the harbor entrance and travels 25 miles in the direction $N 42^\circ E$. The captain then turns the boat 90° and travels another 18 miles in the direction $S 48^\circ E$. At that time, how far is the boat from the harbor entrance, and what is the bearing of the boat from the harbor entrance?

Solution:



Problem 3: (25 points)

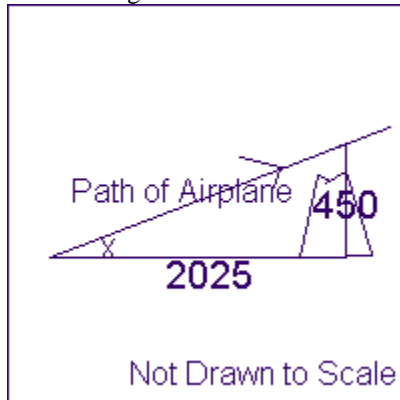
The peak of a Mountain is 400 meters higher than the local airstrip. The horizontal distance from the end of the runway to a point directly below the mountain peak is 2025 m.

A plane takes off at the end of a runway in the direction of the mountain at an angle that is kept constant until the peak has been cleared. If the pilot, Jim, wants to clear the mountain by 50 m, what should be the angle of takeoff to the nearest tenth of a degree?

Solution:

The angle at which the pilot must fly in order to clear the peak at 50 meters, is 12.5 degrees. First, I added 50 to 400, since that is the height that the pilot wishes to pass thru. Making that, the opposite end of a triangle, and 2025 m, the distance the end of the runway is from the mountain, the adjacent or base of the triangle (see drawing), at 90 degrees from 450. Then I drew the hypotenuse, which is the path of the plane, flying at x degrees. To figure out x, I used tangent with the opposite and adjacent.

$$\begin{aligned} \tan x &= \text{opp/adj} \\ \tan x &= 450/2025 \\ x &= \tan^{-1} (450/2025) \\ x &= 12.5 \text{ degrees} \end{aligned}$$



Calendar

Problem 1: (100 points)

What day of the week was March 8th in 1929?

Solution:

7 days ago was a Saturday; $7k$ days ago was a Saturday for any integer k . Therefore 364 days ago was a Saturday, and 1 (regular) year ago was a Friday. Thus, for every regular year, March 8th moves back 1 day of the week. For every leap year, March 8th moves back two days of the week. There are 74 years between 1929 and 2003, 18 of which are leap years (1932, 1936....1996, 2000). Thus March 8th, 1929 occurred $56*1 + 18*2 = 13$ week and 1 day before a Saturday. March 8th 1929 was a Friday.

Problem 2: (50 points)

When tomorrow is yesterday, today will be as near to Sunday as today was when yesterday was tomorrow. What day is it??

Solution:

“Today” is Sunday. Tomorrow will be yesterday when it’s the day after tomorrow. That means two days from now. Similarly, a day when yesterday was still “tomorrow”, had to have been two days ago. So two days from now it was the same number of days from Sunday as it will be in two days time.

Problem 3: (25 points)

Suppose UD wanted to hold the High School Math contest on March 8th in 2022. What day of the week would the contest fall upon?

Solution:

Each regular year “moves” March 8th one day forward. Each leap year moves it two days forward. Thus this would be the table:

Year:	Day of the week March 8 th falls on:
2004	Monday
2005	Tuesday
2006	Wednesday
2007	Thursday
2008	Saturday
2009	Sunday
2010	Monday
2011	Tuesday
2012	Thursday
2013	Friday
2014	Saturday
2015	Sunday
2016	Tuesday
2017	Wednesday
2018	Thursday
2019	Friday

2020	Sunday
2021	Monday
2022	Tuesday

Therefore the day would be Tuesday

Matrices

Let a_{jk} represent the number in the j th row and k th column of an $n \times n$ matrix A . The determinant of a 2×2

matrix is the number: $\det A = a_{11} a_{22} - a_{12} a_{21}$.

For any matrix A , let A_{jk} denote the submatrix formed by deleting the j th row and k th column of A . For $n \geq 2$, the determinant of an $n \times n$ matrix A is the sum of n terms of the form $\pm a_{jk} \det A_{jk}$, with plus and

minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row of A : $\det A =$

$$a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n} = \sum_{k=1}^n (-1)^{1+k} a_{1k} \det A_{1k}.$$

Problem 1: (100 points)

$$\text{Let } A = \begin{vmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{vmatrix}$$

Find $\det A$.

Solution:

$$\det A = 2^4(-158) = 16(-158) = -2528$$

Problem 2: (50 points)

The matrix A represents the average score for each student on homework, quizzes, and test. Homework is 40% of the final grade, quizzes are 20% and tests are 40%.

	H	Q	T	
Kim	90	87	95	= A
Bob	92	90	88	
Dan	83	70	76	
Ann	84	80	78	
Tim	95	91	93	
Joe	76	71	67	

Use matrices to determine the final grade for each student. You must first write the vector B , expressing the percentages that would be used to find the final grade for each student. Then use A and B to determine your answer.

Solution:

$$B = \begin{matrix} .4 & H \\ .2 & Q \\ .4 & T \end{matrix}$$

$$\begin{aligned} AB &= 91.4 \\ &90 \\ &77.6 \\ &80.8 \\ &93.4 \\ &71.4 \end{aligned}$$

Problem 3: (25 points)

$$\text{Let } A = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

Find $\det A$.

$$\text{Let } B = \begin{vmatrix} 1 & 5 & 8 \\ 0 & 0 & -2 \\ 2 & 4 & -4 \end{vmatrix}$$

Use $\det A$ to find $\det B$. Write both $\det A$ and $\det B$.

Solution:

$$\det A = (1)(4) - (5)(2) = 4 - 10 = -6$$

$$\det B = 2 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = (2)(\det A) = 2(-6) = -12$$

Logic

Problem 1: (100 points)

There are three bands in the final round of a Battle of the Bands contest. Each must play two songs throughout the round, but they must go in a certain order. The following guidelines govern the schedule of the final program:

C must be earlier E and earlier than G.

F must be performed later than T and earlier than G.

A must be earlier than C and before T.

Band 1 will perform A and C.

Band 2 will perform E and another song.

Band 3 will perform G and another song.

The band who performs first must be different than the band who performs last.

Not one of the bands perform both of their songs consecutively.

What is the order of the songs and which bands play which songs?

Solution:

A T C F E G

A T F C E G

A T F C E G

Band 1: A and C

Band 1: A and C

Band 1: A and C

Band 2: T and E

Band 2: T and E

Band 2: F and E

Band 3: F and G

Band 3: F and G

Band 3: T and G

A T C F G E

A T F C G E

A T F C G E

Band 1: A and C

Band 1: A and C

Band 1: A and C

Band 2: F and E

Band 2: T and E

Band 2: F and E

Band 3: T and G

Band 3: F and G

Band 3: T and G

Problem 2: (50 points)

After examining a woman's life, it was determined that one sixth of her life was spent as a child. After 7 years of being a teenager, one twelfth of her life passed between the time she became an adult and when she got married. One twenty-fourth of her life later, she gave birth to her daughter. Unfortunately, her daughter passed away thirteen years before she did. She lived to be twice the age of her daughter. How old did the woman live to be?

Solution:

Let x be the age of the woman at the time of her death. Her age is the sum of the years she spent as a child, the years she spent as a teenager, the years between becoming an adult and getting married, the years between getting married and having her daughter, her daughter's age (half of her own), and the thirteen years after her daughter's death. This can also be shown:

$$x = x/6 + 7 + x/12 + x/24 + 13 + x/2$$

$$x = 4x/24 + 2x/24 + x/24 + 12x/24 + 20$$

$$x = 19x/24 + 20$$

$$24x/24 - 19x/24 = 20$$

$$5x/24 = 20$$

$$x = 96$$

Therefore, the woman lived to be 96 years old.

Problem 3: (25 points)

In the addition problem below, find the value of each letter that makes the addition correct. Each integer, 0 – 9, is represented by one letter. Give your answer as a listing of the values 0 – 9.

$$\begin{array}{r} \text{MULTIPLY} \\ + \text{DIVIDE} \\ \hline \text{MUMIYDPP} \end{array}$$

Solution:

0	1	2	3	4	5	6	7	8	9
T	D	U	V	I	L	M	P	Y	E

Miscellaneous

Problem 1: (100 points)

How many ways can you uniquely rearrange these letters assuming that the words stay together, and that the spelling of one word doesn't affect the others (ignoring punctuation)?

Hence:

If you have the phrase "noon noon" you will have to distinguish that as 2 different rearrangements. While the spelling noon for each word would only be counted once.

The University of Dayton's High School Math Challenge

Solution:

Spellings of "The" = $3! = 6$

Spellings of "University" = $10!/(2(a's)) = 3628800/2 = 1814400$

Spellings of "of" = $2! = 2$

Spellings of "Dayton's" = $7! = 5040$

Spellings of "High" = $4!/(2(h's)) = 24/2 = 12$

Spellings of "School" = $6!/(2(o's)) = 720/2 = 360$

Spellings of "Math" = $4! = 24$

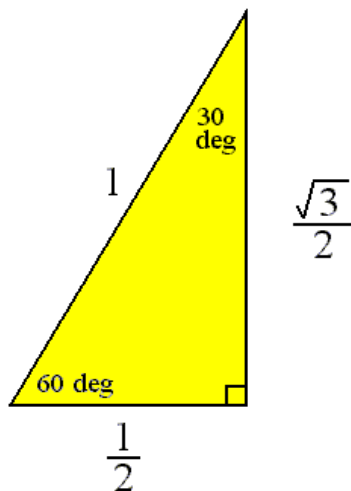
Spellings of "Challenge" = $9!(2(l's)*2(e's)) = 362880/4 = 90720$

$$6 * 1814400 * 2 * 5040 * 12 * 360 * 24 * 90720 = 1032150078141235200000$$

$$1032150078141235200000 * 8!(\text{was the words can be ordered}) = 41616291150654603264000000 \\ = 4.161629 * 10^{25}$$

Problem 2: (50 points)

A baseball player hits a ball off of a bat at a 30 degree angle 5 ft off the ground, with out the limitation of gravity how high is the ball after it travels a horizontal distance of 433 feet (hence the distance traveled is if the ball was hit at a 0 degree angle). Round your answer to the nearest foot.
Hint:



Solution:

$$\frac{433}{\sqrt{3}} \approx 250 \quad 250 + 5 = 255\text{ft.}$$

Problem 3: (25 points)

Choose any 2-digit number, multiply it by 3. now add 15. multiply by 2. subtract 6. Divide by 6. Subtract the original number with the digits reversed. Add 32. Divide by 9. Add the original number's units digit and subtract the tens digit. What will the result always be?

Solution:

4

Probability and Statistics

Problem 1: (100 points)

Bart is planning to murder his rich uncle Basil in hopes of claiming his inheritance a bit early. Hoping to take advantage of his uncle predilection for immoderate desserts, Bart has put rat poison in the cherries flambé and cyanide in the chocolate mousse. The probability of the rat poison being fatal is 0.6, that of the cyanide killing someone is 0.9. Based on other dinners he has had with his uncle, Bart estimates that Basil has a 60% chance of asking for cherries flambé and a 40% chance of ordering the chocolate mousse. Given that Basil did, indeed suffer a premature demise, what is the probability it was the chocolate mousse that did him in?

Solution:

Solution: We know:

$$P(\text{chooses Flambé}) = 0.6; P(\text{dies} \mid \text{chooses Flambé}) = 0.6$$

$$P(\text{chooses Mousse}) = 0.4; P(\text{dies} \mid \text{chooses Mousse}) = 0.9$$

Thus

$$P(\text{chooses Mousse} \mid \text{dies}) = \frac{P(\text{chooses Mousse and dies})}{P(\text{dies})}$$

$$= \frac{P(\text{dies} \mid \text{chooses Mousse}) P(\text{chooses Mousse})}{P(\text{dies} \mid \text{chooses Mousse}) P(\text{chooses Mousse}) + P(\text{dies} \mid \text{chooses Flambé}) P(\text{chooses Flambé})}$$

$$= \frac{(0.9)(0.4)}{(0.9)(0.4) + (0.6)(0.6)} = 1/2$$

Problem 2: (50 points)

All students at High School A and High School B take a certain exam. The average scores for boys, for girls, and for boys and girls combined, at High School A and High School B are shown in the table, as is the average for boys at the two schools combined. What is the average score for the girls at the two schools combined?

	High School A	High School B	High School A&B
Boys	71	81	79
Girls	76	90	?
Boys & Girls	74	84	

Solution:

We want a weighted average, X of 76 and 90, with weights proportional to the number of girls at High School A and High School B, respectively. We obtain these weights as follows: Let

b=number of boys at High School A

g=number of girls at High School A

B=number of boys at High School B

G=number of girls at High School B

From the first column of the table obtain

$$\frac{71b + 76g}{b + g} = 74$$

$$b + g$$

So we solve $71b + 76g = 74b + 74g$ to find that $g = 1.5b$. Similarly, the second column shows that $G = 1.5B$ and the first row shows that $B = 4b$. Thus

$$X = \frac{76g + 90G}{g + G} = \frac{76(1.5b) + 90[1.5(4b)]}{1.5b + [1.5(4b)]} = \frac{114 + 180}{3.5} = 84$$

Problem 3: (25 points)

Horses X, Y, and Z are entered in a three-horse race in which ties are not possible. If the odds against X winning are 3-to-1 and the odds against Y winning are 2-to-3, what are the odds against Z winning? (By “odds against H winning are p-to-q” we mean the probability of H winning the race is $\frac{q}{p+q}$)

$$\frac{q}{p+q}$$

Solution:

The probability that X wins is $\frac{1}{3+1}$ and the probability that Y wins is $\frac{3}{2+3}$. The sum of the winning probabilities for all three horses must be 1, so the probability that Z wins is

$$1 - \frac{1}{4} - \frac{3}{5} = \frac{3}{20} = \frac{3}{17+3}$$

Hence the odds against Z winning are 17-to-3