BOUNDDEDNESS IN NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. Non-negative definite Lyapunov functions are employed to obtain sufficient conditions that guarantee boundedness of solutions of a nonlinear differential system. The theory is illustrated with several examples.

1. Introduction

In this paper, we make use of non-negative definite Lyapunov functions and obtain sufficient conditions that guarantee the boundedness of all solutions of the initial value problem

\[\begin{align*}
\dot{x} &= f(t, x), \quad t \geq 0, \\
x(t_0) &= x_0, \quad t_0 \geq 0
\end{align*}\]

(1.1)

where \(x \in \mathbb{R}^n, f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n\) is a given nonlinear continuous function in \(t\) and \(x\), where \(t \in \mathbb{R}^+\). Here \(\mathbb{R}^n\) is the \(n\)-dimensional Euclidean vector space; \(\mathbb{R}^+\) is the set of all non-negative real numbers; \(\|x\|\) is the Euclidean norm of a vector \(x \in \mathbb{R}^n\). Our interest in studying the boundedness of solutions of (1.1) arises from the fact that in the study of the stability of the zero solution of (1.1) when \(f(t, 0) = 0\), one may have to assume the existence of solutions for all positive time. For more on the stability, we refer the interested reader to [1] and [6]. In [2], the authors obtained, using positive definite Lyapunov functionals, interesting results regarding the boundedness of solutions of systems that are similar to (1.1). For more results regarding the stability of the zero solution or boundedness of solutions of (1.1), we refer the interested reader to [4], [5], [7] and [9]. In the spirit of the work in [3] and [6], in this investigation, we establish sufficient conditions that yield all solutions of (1.1) are bounded. We achieve this by assuming the existence of a Lyapunov function that is bounded below and above and that its derivative along the trajectories of (1.1) to be bounded by a negative definite function, plus a positive constant.

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2. **Boundedness of Solutions**

In this section we use non-negative Lyapunov type functions and establish sufficient conditions to obtain boundedness results on all solutions \(x(t)\) of (1.1). From this point forward, if a function is written without its argument, then the argument is assumed to be \(t\).

**Definition 2.1** We say that solutions of system (1.1) are bounded, if any solution \(x(t, t_0, x_0)\) of (1.1) satisfies

\[
||x(t, t_0, x_0)|| \leq C(||x_0||, t_0), \quad \text{for all } t \geq t_0,
\]

where \(C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+\) is a constant that depends on \(t_0\) and \(x_0\). We say that solutions of system (1.1) are uniformly bounded if \(C\) is independent of \(t_0\).

If \(x(t)\) is any solution of system (1.1), then for a continuously differentiable function

\[
V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+,
\]

we define the derivative \(V'\) of \(V\) by

\[
V'(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, x)}{\partial x_i} f_i(t, x).
\]

**Theorem 2.2** Let \(D\) be a set in \(\mathbb{R}^n\). Suppose there exist a continuously differentiable Lyapunov function \(V : \mathbb{R}^+ \times D \to \mathbb{R}^+\) that satisfies

\[
\lambda_1 ||x||^p \leq V(t, x) \leq \lambda_2 ||x||^q,
\]

and

\[
V'(t, x) \leq -\lambda_3 ||x||^r + L\tag{2.2}
\]

for some positive constants \(\lambda_1, \lambda_2, \lambda_3, p, q, r\) and \(L\). Moreover, if for some constant \(\gamma \geq 0\) the inequality

\[
V(t, x) - V^{r/q}(t, x) \leq \gamma\tag{2.3}
\]

holds, then all solutions of (1.1) that stay in \(D\) are uniformly bounded.

**Proof** Let \(M = \lambda_3/\lambda_2^{r/q}\). For any initial time \(t_0\), let \(x(t)\) be any solution of (1.1) with \(x(t_0) = x_0\). Then,

\[
\frac{d}{dt} (V(t, x(t))e^{M(t-t_0)}) = \left[V'(t, x(t)) + MV(t, x(t))\right]e^{M(t-t_0)}.
\]
For $x(t) \in \mathbb{R}^n$, using (2.2) we get
\[
\frac{d}{dt} \left( V(t, x(t)) e^{M(t-t_0)} \right) \leq \left[ -\lambda_3 ||x||^r + L + MV(t, x(t)) \right] e^{M(t-t_0)}, \tag{2.4}
\]
From condition (2.1) we have $||x||^q \geq V(t, x)/\lambda_2$.

Consequently, $-||x||^r \leq -\frac{V(t, x(t))}{\lambda_2^{r/q}}$. In addition, if we use (2.3) inequality (2.4) becomes
\[
\frac{d}{dt} \left( V(t, x(t)) e^{M(t-t_0)} \right) \leq \left[ \frac{-\lambda_3}{\lambda_2^{r/q}} V(t, x(t)) + MV(t, x(t)) \right] e^{M(t-t_0)} = \left( M \gamma + L \right) e^{M(t-t_0)} = : Ke^{M(t-t_0)}.
\]
Integrating the above inequality from $t_0$ to $t$ we obtain,
\[
V(t, x(t)) e^{M(t-t_0)} \leq V(t_0, x_0) + \frac{K}{M} e^{M(t-t_0)} - \frac{K}{M} \leq \lambda_2 ||x_0||^q + \frac{K}{M} e^{M(t-t_0)}.
\]
Consequently,
\[
V(t, x(t)) \leq \lambda_2 ||x_0||^q e^{-M(t-t_0)} + \frac{K}{M}.
\]
From condition (2.1) we have $\lambda_1 ||x||^p \leq V(t, x(t))$, which implies that
\[
||x|| \leq \left\{ \frac{1}{\lambda_1} \right\}^{1/p} \left[ \lambda_2 ||x_0||^q + \frac{K}{M} \right]^{1/2} \text{ for all } t \geq t_0.
\]
This completes the proof. \hfill \diamond

**Example 2.3** Consider the semi-linear differential equation
\[
x' = \sigma x + Rx^{1/3}. \tag{2.5}
\]
If
\[
2\sigma + \frac{4}{3} = -\alpha \quad \text{for some } \alpha > 0,
\]
then all solutions of (2.5) are uniformly bounded.

To see this, let $V(t, x) = x^2$. Then along solutions of (2.5) we have

$$V'(t, x) = 2xx' = 2\sigma x^2 + 2Rx^{4/3} \leq 2\sigma x^2 + 2|R|x^{4/3}. \quad (2.6)$$

To further simplify (2.6), we make use of Young’s inequality, which says for any two nonnegative real numbers $w$ and $z$, we have

$$wz \leq \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } 1/e + 1/f = 1.$$

Thus, for $e = 3/2$ and $f = 3$, we get

$$2|R|x^{4/3} \leq 2\left[\frac{1}{3}|R|^3 + \frac{(x^{4/3})^{3/2}}{3/2}\right] = \frac{4}{3}x^2 + \frac{4}{3}|R|^3.$$

By substituting the above inequality into (2.6), we arrive at

$$V'(t, x) \leq \left(2\sigma + \frac{4}{3}\right)x^2 + \frac{2}{3}|R|^3 = -\alpha x^2 + \frac{2}{3}|R|^3.$$

One can easily check that conditions (2.1)-(2.3) of Theorem 2.2 are satisfied with $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \alpha$, $L = \frac{2}{3}|R|^3$ and $p = q = r = 2$. Hence all solutions of (2.5) are uniformly bounded. ◦

In Example 2.3, condition (2.3) did not come into play, which was due to the fact that $r = q = 2$. In the next example, we consider a nonlinear system in which condition (2.3) naturally comes into play.

**Example 2.4** Let $D = \{ x \in \mathbb{R} : ||x|| \geq 1 \}$ and consider the nonlinear differential equation

$$x' = -\frac{5}{6}x^3 + Rx^{1/3}, \quad t \geq 0,$$

$$x(0) = 1.$$
Consider the Lyapunov functional $V(t, x) : \mathbb{R}^+ \times D \rightarrow \mathbb{R}^+$ such that $V(t, x) = x^2$. Then along solutions of the differential equation we have

\[
V' = 2xx' \\
= -\frac{5}{3}x^4 + 2Rx^{4/3} \\
\leq -\frac{5}{3}x^4 + 2|R|x^{4/3}
\]  

(2.8)

Using Young’s inequality with $e = 3$ and $f = 3/2$, we get

\[
|x|^{4/3}|R| \leq \frac{x^4}{3} + \frac{2}{3}|R|^{3/2}.
\]

Hence

\[
V'(t, x) \leq -x^4 + \frac{4}{3}|R|^{3/2}.
\]

We take $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 1$, $L = \frac{4}{3}|R|^{3/2}$, $p = q = 2$ and $r = 4$.

For $x \in D$

\[
V(t, x) - V^{r/q}(t, x) = x^2(1 - x^2) \leq 0.
\]

Thus, condition (2.3) is satisfied for $\gamma = 0$. An application of Theorem 2.2 yields

\[
|x(t)| \leq \left\{ \frac{1}{\lambda_1} \right\}^{1/p} \left[ \lambda_2 ||x_0||^q + \frac{K \gamma^{1/p}}{M} \right] \\
= \left( 1 + \frac{L}{M} \right)^{1/2} \\
= \left( 1 + \frac{4}{3}|R|^{3/2} \right)^{1/2} \text{ for all } t \geq 0.
\]

Hence, every solution $x$ with $x(t) \in D$ satisfies

\[
1 \leq |x(t)| \leq \left( 1 + \frac{4}{3}|R|^{3/2} \right)^{1/2}, \text{ for } t \geq 0.
\]

\[\square\]

In the next theorem we show that solutions of (1.1) are bounded.

**Theorem 2.5** Suppose there exist a continuously differentiable Lyapunov function $V : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ that satisfies

\[
\lambda_1(t)||x||^p \leq V(t, x) \leq \lambda_2(t)||x||^q,
\]

(2.9)

and

\[
V'(t, x) \leq -\lambda_3(t)||x||^r + L
\]

(2.10)

for some positive constants $p, q, r, L$, and positive continuous functions $\lambda_1(t), \lambda_2(t)$, and $\lambda_3(t)$, where $\lambda_1(t)$ is nondecreasing. Moreover, if for some constant $\gamma \geq 0$ the inequality
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\[ V(t, x) - V^{r/q}(t, x) \leq \gamma \quad (2.11) \]
holds, then all solutions of (1.1) are bounded.

**Proof** Let

\[ M = \inf_{t \in \mathbb{R}^+} \frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}}. \]

For any initial time \( t_0 \), let \( x(t) \) be any solution of (1.1) with \( x(t_0) = x_0 \).

By calculating

\[ \frac{d}{dt} \left( V(t, x(t))e^{M(t-t_0)} \right) \]
and then by a similar argument as in Theorem 2.2 we obtain,

\[ V(t, x(t)) \leq \lambda_2(t_0)||x_0||^q e^{-M(t-t_0)} + \frac{K}{M}. \quad (2.12) \]

Since \( \lambda_1(t) \) is nondecreasing and by using (2.7), we arrive at

\[ ||x|| \leq \left\{ \frac{V(t, x(t))}{\lambda_1(t)} \right\}^{1/p} \leq \left\{ \frac{V(t, x(t))}{\lambda_1(t_0)} \right\}^{1/p}. \quad (2.13) \]

Combining (2.10) and (2.11) we get

\[ ||x|| \leq \left\{ \frac{1}{\lambda_1(t_0)} \right\}^{1/p} \left[ \lambda_2(t_0)||x_0||^q + \frac{K}{M} \right]^{1/p} \text{ for all } t \geq t_0. \]

This completes the proof. 

The next theorem does not require an upper bound on the Lyapunov function.

**Theorem 2.6** Suppose there exist a continuously differentiable Lyapunov function \( V: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+ \) that satisfies

\[ \lambda_1||x||^p \leq V(t, x), \quad (2.14) \]
and

\[ V'(t, x) \leq -\lambda_2 V(t, x) + L \quad (2.15) \]
for some positive constants \( \lambda_1, \lambda_2, p \) and \( L \).

Then, all solutions of (1.1) are bounded.

**Proof** For any initial time \( t_0 \), let \( x(t) \) be any solution of (1.1) with \( x(t_0) = x_0 \). By calculating

\[ \frac{d}{dt} \left( V(t, x(t))e^{\lambda_2 t} \right) \]
and making use of (2.15) we obtain
\[
\frac{d}{dt} \left( V(t, x)e^{\lambda_2 t} \right) \leq L e^{\lambda_2 t}
\]

An integration of the above inequality from \( t_0 \) to \( t \) yields

\[
V(t, x) \leq V(t_0, x_0) e^{-\lambda_2 (t-t_0)} + \frac{L}{\lambda_2}.
\]

Using (2.14) into the above inequality, we have

\[
||x|| \leq \left\{ \frac{1}{\lambda_1} \right\}^{1/p} \left[ V(t_0, x_0) + \frac{L}{\lambda_2} \right]^{\frac{1}{p}} \text{ for all } t \geq t_0.
\]

This completes the proof. \( \diamond \)

In the next theorem our Lyapunov function satisfies a different type of condition on its derivative along the solutions.

**Theorem 2.7** Suppose there exist a continuously differentiable Lyapunov function \( V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+ \) that satisfies

\[
\lambda_1 ||x||^p \leq V(t, x), \quad V(t, x) \neq 0 \text{ if } x \neq 0 \quad (2.16)
\]

and

\[
V'(t, x) \leq -\lambda_2(t)V^q(t, x) \quad (2.17)
\]

for some positive constants \( \lambda_1, p, q > 1 \) where \( \lambda_2(t) \) is a positive continuous function such that

\[
c_1 = \inf_{t \geq t_0 \geq 0} \lambda_2(t) > 0. \quad (2.18)
\]

Then all solutions of (1.1) are bounded.

**Proof** For any initial time \( t_0 \geq 0 \), let \( x(t) \) be any solution of (1.1) with \( x(t_0) = x_0 \). By calculating

\[
\frac{d}{dt} V(t, x(t))
\]

along the solutions of (1.1) and making use of (2.17) and (2.18), we obtain

\[
V'(t, x(t)) \leq -\lambda_2(t)V^q(t, x(t)) \leq -c_1 V^q(t, x(t)).
\]

Denoting \( u(t) = V(t, x(t)) \), we have that this function satisfies the following differential inequality

\[
\dot{u}(t) \leq -c_1 [u(t)]^q,
\]

and, therefore its positive solutions satisfy

\[
\frac{\dot{u}(t)}{[u(t)]^q} \leq -c_1.
\]
An integration of the above inequality from $t_0$ to $t$ yields

$$V(t, x(t)) \leq \left[ V^{1-q}(t_0, x_0) + c_1(q - 1)(t - t_0) \right]^{-1/(q-1)}.$$

By invoking condition (2.16) we arrive at

$$||x|| \leq 1/\lambda_1^{1/p}\left\{ \left[ V^{1-q}(t_0, x_0) + c_1(q - 1)(t - t_0) \right]^{-1/(q-1)} \right\}^{1/p}.$$

This completes the proof.

As an application of the previous theorem, we furnish the following example.

**Example 2.8** To illustrate the application of Theorem 2.7, we consider the following two dimensional system

$$y'_1 = y_2 - y_1|y_1| - \frac{y_2}{1 + y_1^2}$$
$$y'_2 = -y_1 - y_2|y_2| + \frac{y_1}{1 + y_2^2}$$
$$y_1(t_0) = y_1(0), \quad y_2(0) = y_2(0) \text{ for } t_0 \geq 0.$$

Let us take $V(t, y_1, y_2) = \frac{1}{2}(y_1^2 + y_2^2)$. Then

$$V'(t, y_1, y_2) = -y_1^2|y_1| - y_2^2|y_2| - \frac{y_1 y_2}{1 + y_1^2} - \frac{y_2 y_1}{1 + y_2^2}$$
$$= -\left( |y_1|^3 + |y_2|^3 \right)$$
$$= -2\left[ \frac{|y_1|^3}{2} + \frac{|y_2|^3}{2} \right]$$
$$= -2\left[ \left( \frac{|y_1|^2}{2} \right)^{3/2} + \left( \frac{|y_2|^2}{2} \right)^{3/2} \right]$$
$$\leq -2 \left( |y_1|^2 + |y_2|^2 \right)^{3/2} 2^{-3/2}$$
$$= -2V^{3/2}(t, y_1, y_2)$$

where we have used the inequality $\left( \frac{a+b}{2} \right)^l \leq \frac{a^l}{2} + \frac{b^l}{2}$, $a, b > 0$, $l > 1$. Hence, by Theorem 2.7 all solutions of the above two dimensional system are uniformly bounded.

**References**


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