Box-Hunter resolution extends to arbitrary fractional factorial designs

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Abstract

In a 1961 paper, Box and Hunter defined the resolution of a regular fractional factorial design as a measure of the amount of aliasing in the fraction. They indicated that the maximum resolution is equal to the minimum length of a defining word.

Since then, various approaches have been offered to generalize the concept of resolution to arbitrary (possibly mixed-level) fractions. These have generally been based on estimability and on the assumption that high-order interactions are absent, rather than on the alias structure of the fraction. On the other hand, it is not hard to formulate a generalization of Box-Hunter resolution based on an idea that may be traced back to Rao (1947). Using it, we show that in an arbitrary fraction of maximum strength $t$ and maximum resolution $R$, we have $R = t + 1$.

For completeness, we include a proof that in regular fractions of strength $t$, the minimum wordlength is also $t + 1$.

1 Introduction: aliasing and resolution

Consider the following example, familiar from many introductory texts on experimental design:

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Columns A through ABC give the orthogonal contrasts representing each effect in the experiment. As usual, we note the following:

- If one limits attention to the four cells in the “fraction” above the center line and to the corresponding columns of length 4, then certain pairs of contrasts are equal, or “aliased,” in the fraction, others remain orthogonal (“unaliased”), and one column is no longer a contrast at all.

- Within the fraction the main effects are not aliased with each other, but each is aliased with a two-factor interaction. Such a fractional design is said to have resolution 3.

Box and Hunter introduced this notion of resolution [6, page 319] to formalize something easily observed in so-called regular fractions:

There is an integer \( R \) such that no interaction of \( p \) factors is aliased with any interaction of fewer than \( R - p \) factors.

The number \( R \) thus acts as a kind of fulcrum, and is the resolution of the design.

Box and Hunter also noted that the resolution of such a design is easily calculated as the length of the shortest “defining word.” (The defining word of the fraction above is \( ABC \).) It is not uncommon to define resolution as minimum wordlength. Significantly, however, wordlength does not appear in their definition of resolution, but only in a remark following it. Whether or not the omission was intended, it naturally tempts us to extend the definition to nonregular, possibly mixed-level fractions. In order to make this work, we must make precise the definition of “aliased” (and, more to the point, of “unaliased”). That, in turn, requires that we specify just what we mean by a main effect or interaction in an arbitrary fraction.

The most direct answer is to define effects in precisely the same way that we do in the example above: we simply restrict all relevant contrasts to the treatment combinations appearing in the fraction (see equation (4) below). Having done that, we then define two effects to be aliased in the fraction if the corresponding sets of restricted contrasts are equal, and unaliased if they are mutually orthogonal. This allows us [3] to define resolution in an arbitrary fraction exactly as Box and Hunter did (Definition 2.4 below), without assuming regularity of the fraction at all.

A fractional factorial design has a certain strength when viewed as an orthogonal array. As is well known, if a regular fraction has minimum wordlength \( w \) and maximum strength \( t \), then \( w = t + 1 \). Thus strength serves as a proxy for wordlength in non-regular fractional designs. In
a previous paper [3] it has been shown that an arbitrary fraction of strength \( t \) has Box-Hunter resolution \( R \geq t + 1 \) (Corollary 2.6 below). In this paper we show that if the maximum strength of the fraction is \( t \) and its maximum resolution is \( R \), then \( R = t + 1 \) (Theorem 2.7). This is the main result of the present paper, and it generalizes the relationship between resolution and wordlength to arbitrary fractions.

Our overall approach rests on the same orthogonality argument (Theorem 2.5) that leads to Rao’s inequalities for mixed-level orthogonal arrays. In fact, the present definition of effects in an arbitrary fractional design is not only the same as is used in the Box-Hunter approach, but is also precisely that given by Rao [15] when he introduced the concept of strength for symmetric fractional designs. The approach leading to Theorem 2.7 is also greatly influenced by the work of Tjur [16].

For completeness, we include a proof of the fact, mentioned above, that \( w = t + 1 \) in a regular fraction (Corollary A.3). The proof rests on an argument concerning the smallest number of non-zero coefficients of any defining equation (Theorem A.2). The latter result is essentially equivalent to Theorem 4.6 of [12], but it avoids the use of coding theory, and its proof differs in certain essentials. Theorem 8.2.2 of [9] gives one side of this result. (The latter authors state the result in terms of resolution, but they use the word “resolution” as a synonym for wordlength, as we note in Section A.)

Other approaches to resolution and aliasing will be discussed in Section 3.

**Concerning “maximum” strength and “maximum” resolution.** The usual definition of strength is such that if a fraction has strength \( t \) then it automatically has strength \( t' \) for all \( t' < t \) as well. Similarly, if we use the above definition of resolution, then a fraction of resolution \( R \) automatically has resolution \( R' \) for all \( R' < R \). For practical purposes, then, we wish to know the maximum strength and maximum resolution of a fraction. This is implicit in Box and Hunter’s description of resolutions 3, 4 and 5. It is the maximum resolution that is equal to the minimum wordlength in regular fractions.

**Notation and basic definitions.** We follow the notation and definitions given in [3]. In particular, the cardinality of a set \( E \) is denoted by \( |E| \), and the empty set by \( \emptyset \). The integers are denoted by \( \mathbb{Z} \), and the integers modulo \( n \) by \( \mathbb{Z}/n \). The real numbers are denoted by \( \mathbb{R} \), and the real-valued functions on the set \( T \) by \( \mathbb{R}^T \). Given any finite set \( T \) (for us, the set of treatment combinations), \( \mathbb{R}^T \) is a Euclidean space with inner product

\[
(u, v) = \sum_{s \in T} u(s)v(s)
\]

for \( u, v \in \mathbb{R}^T \) and norm \( ||v|| = \sqrt{(v, v)} \). If we fix an ordering of the elements of \( T \), we may view \( u \) and \( v \) as ordinary column vectors in the Euclidean space \( \mathbb{R}^g \), where \( g = |T| \). Then the formula in (1) is the ordinary dot product.

We denote by \( 1 \) the constant function taking the value 1, and by \( 1_C \) the indicator or characteristic function of the set \( C \subset T \):

\[
1_C(s) = \begin{cases} 
1 & \text{if } s \in C, \\
0 & \text{if } s \notin C.
\end{cases}
\]

Thus \( 1 \) is \( 1_T \). Note that \((1_C, 1_D) = |C \cap D|\).
If there are \( k \) factors whose levels are indexed by sets \( A_1, \ldots, A_k \) of size \( s_1, \ldots, s_k \), respectively, then the set of treatment combinations (or \textit{cells}) is \( T = A_1 \times \cdots \times A_k \). We will refer to \( T \) as the \textit{full factorial design}, and a subset \( S \) of \( T \) will be called a \textit{fractional factorial design} or \textit{fraction}. (The term “fraction” is sometimes applied to allow repeated cells – see [14, page 10] and [9, page 11], for example. However, in this paper we reserve it for designs in which all cells are distinct.)

The design \( T \) is \textit{symmetric} if \( s_1 = \cdots = s_k = s \), in which case we may take \( A_1 = \cdots = A_k = A \); otherwise it is \textit{asymmetric} or \textit{mixed-level}. Similar terminology applies to a fraction. If in a symmetric design \( s \) is a prime or prime power, we may take \( A \) to be the finite field \( \text{GF}(s) \).

In this case the fraction is \textit{regular} if it is the solution set of a system of linear equations over the finite field \( \text{GF}(s) \).

If the cells of the fraction are written as rows or columns of a matrix, then the fraction is an \textit{orthogonal array} and thus has strength \( t \), for some \( t \), and (in the symmetric case) index \( \lambda \) (see Section 2).

Other notation is introduced as needed.

2 \textbf{Strength and resolution of fractional factorial designs}

Let \( T \) be a finite set – for us, a set of treatments. An observation on a treatment \( s \in T \) is assumed to have a mean \( \mu(s) \), which we refer to as a \textit{cell mean} (when \( T \) is a Cartesian product, its elements are “cells”). Contrasts in cell means are expressions of the form

\[
\sum_{s \in T} c(s)\mu(s)
\]

where \( \sum_{s \in T} c(s) = 0 \). We may refer to these functions \( c \in \mathbb{R}^T \) as \textit{contrast functions} or \textit{contrast vectors}, or (by abuse of language) as \textit{contrasts}.

Any blocking (or partition) \( \mathcal{C} \) of \( T \) determines a subspace \( U_{\mathcal{C}} \subset \mathbb{R}^T \) of dimension \( |\mathcal{C}| - 1 \) consisting of the contrast functions that are constant on the blocks of \( \mathcal{C} \). If \( c \in U_{\mathcal{C}} \), then \( \sum c(t)\mu(t) \) is a contrast between the blocks. The association of a vector space \( U_{\mathcal{C}} \) to each partition \( \mathcal{C} \) was first formalized and studied by Tjur [16].

If \( \mathcal{D} \) is another blocking of \( T \), we define the \textit{join} of \( \mathcal{C} \) and \( \mathcal{D} \) to be the partition

\[
\mathcal{C} \vee \mathcal{D} = \{ C \cap D : C \in \mathcal{C}, D \in \mathcal{D}, C \cap D \neq \emptyset \}.
\]

Let \( \pi \) be the uniform probability measure on \( T \):

\[
\pi(A) = |A|/|T|.
\]

We denote the independence of \( A \) and \( B \) by \( A \perp \! \! \! \perp B \). This is simply the combinatorial condition

\[
|A \cap B|/|T| = |A||B|.
\]

We say that the set \( A \) is independent of the partition \( \mathcal{C} \) (written \( A \perp \! \! \! \perp C \)) if \( A \perp C \) for every \( C \in \mathcal{C} \). Similarly, the partitions \( \mathcal{C} \) and \( \mathcal{D} \) are independent (\( \mathcal{C} \perp \! \! \! \perp \mathcal{D} \)) if \( C \perp D \) for every \( C \in \mathcal{C} \) and \( D \in \mathcal{D} \). This condition is important because of the fact [1, Lemma 3] that

\[
U_{\mathcal{C}} \perp \! \! \! \perp U_{\mathcal{D}} \text{ iff } \mathcal{C} \perp \! \! \! \perp \mathcal{D}.
\]
Independence also gives us a convenient way to define the strength of an orthogonal array (see Lemma 2.3 below).

For the remainder of this section, let $T = A_1 \times \cdots \times A_k$ be the set of treatment combinations in an $s_1 \times \cdots \times s_k$ factorial, where $A_i$ indexes the levels of factor $i$ and $s_i = |A_i|$. Which main effect or interaction a contrast belongs to is determined entirely by the coefficients $c(s)$.

As $r$ ranges over $A_i$, the sets
\[ A_1 \times \cdots \times A_{i-1} \times \{r\} \times A_{i+1} \times \cdots \times A_k \]
form a blocking $A_i$ of $T$ consisting of $s_i$ blocks of equal size. For $i < j$ the blocks of $A_i \lor A_j$ are sets of the form
\[ A_1 \times \cdots \times A_{i-1} \times \{r\} \times A_{i+1} \times \cdots \times A_{j-1} \times \{s\} \times A_{j+1} \times \cdots \times A_k \]
where $r \in A_i$ and $s \in A_j$. In general, for any nonempty subset $I \subset \{1, \ldots, k\}$ the factors $i \in I$ determine the blocking $\lor_{i \in I} A_i$ of $T$. Its blocks are formed by taking intersections of blocks, one from each $A_i$, $i \in I$, and are subsets of $T$ of the form $B_1 \times \cdots \times B_k$, where for fixed elements $r_i \in A_i$ we have
\begin{equation}
B_i = \begin{cases} \{r_i\} & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}
\end{equation}

We pause to record some simple observations that will be needed below. Let $\mathcal{C}_I$ denote $\lor_{i \in I} A_i$.

**Lemma 2.1.**

a. $\pi(B) = \frac{1}{\prod_{i \in I} s_i}$ for every block $B \in \mathcal{C}_I$.

b. $\mathcal{C}_I \lor \mathcal{C}_J = \mathcal{C}_{I \cup J}$.

c. $\mathcal{C}_I \lor \mathcal{C}_J \iff I \cap J = \emptyset$.

**Proof.** For $B \in \mathcal{C}_I$, $|B| = \prod_{i=1}^k |B_i| = \prod_{i \notin I} s_i$. Thus $\pi(B) = \prod_{i \notin I} s_i / |T| = \prod_{i \in I} s_i$. This proves (a).

To prove (b), let $B' \in \mathcal{C}_I$ and $B'' \in \mathcal{C}_J$. Then $B' = B'_1 \times \cdots \times B'_k$ and $B'' = B''_1 \times \cdots \times B''_k$ where $B'_i$ is of form (2) and $B''_i$ is of the same form with $I$ replaced by $J$ (and possibly different elements $r_i$). We must show that either $B' \lor B''$ is also of this form, $I \cup J$ replacing $I$, or $B' \lor B'' = \emptyset$. But the first case occurs if $B'_i$ and $B''_i$ agree for all $i \in I \cap J$ (trivially if $I \cap J = \emptyset$), while the second occurs if they disagree. Thus $\mathcal{C}_I \lor \mathcal{C}_J \subset \mathcal{C}_{I \cup J}$.

Conversely, if $B \in \mathcal{C}_{I \cup J}$ then $B$ is of form (2) with $I \cup J$ replacing $I$. Using the given values of $r_i, i \in I \cup J$, define
\begin{equation}
B'_i = \begin{cases} \{r_i\} & \text{if } i \in I, \\ A_i & \text{if } i \notin I. \end{cases}
\end{equation}

and
\begin{equation}
B''_i = \begin{cases} \{r_i\} & \text{if } i \in J, \\ A_i & \text{if } i \notin J \end{cases}
\end{equation}
and put $B' = B'_1 \times \cdots \times B'_k$ and $B'' = B''_1 \times \cdots \times B''_k$. Note that $B'$ and $B''$ automatically agree on $I \cap J$, and that $B' \in \mathcal{C}_I$ and $B'' \in \mathcal{C}_J$. Then $B = B' \lor B'' \in \mathcal{C}_I \lor \mathcal{C}_J$, and so $\mathcal{C}_{I \cup J} \subset \mathcal{C}_I \lor \mathcal{C}_J$, proving (b).
Finally, let $B' \in \mathcal{C}_I$ and $B'' \in \mathcal{C}_J$. If $I \cap J = \emptyset$, then it is easy to see that $\pi(B' \cap B'') = \pi(B') \pi(B'')$. If, however, $I \cap J \neq \emptyset$, then either there exists $i \in I \cap J$ such that $r_i' \neq r_i''$, in which case $\pi(B' \cap B'') = 0$, or $\pi(B' \cap B'') = 1 / \prod_{t \in I \cup J} s_t$. In either case, $\pi(B' \cap B'') \neq \pi(B') \pi(B'')$ for $B' \in \mathcal{C}_I$ and $B'' \in \mathcal{C}_J$. This proves (c).

We now describe the contrasts belonging to main effects and to various interactions in the factorial experiment. First, the contrasts between the blocks of $A_i$ define the main effect of factor $i$. The set of such contrast functions is then

$$U_i = U_{A_i}.$$  

The contrast functions belonging to the $ij$-interaction are defined to be the elements of $U_{A_i \lor A_j}$ that are orthogonal to both $U_i$ and $U_j$. They form a subspace which we denote $U_{ij}$. In general, for $\emptyset \neq I \subset \{1, \ldots, k\}$ we define the subspaces $U_I$ inductively as

$$U_I = \{c \in U_\mathcal{C} : c \perp U_J \text{ for all } J \subsetneq I\},$$

where $\mathcal{C} = \lor_{i \in I} A_i$ and $U_\emptyset$ is the subspace of constant functions. For nonempty $I$, the subspace $U_I$ is the set of contrast functions belonging to the interaction between the factors listed in the set $I$. This is a slightly modernized version of the definition given by Bose [5]. We note that $U_\mathcal{C}$ has the orthogonal decomposition

$$U_\mathcal{C} = \oplus_{J \subseteq I} U_J. \tag{3}$$

If we only observe a fraction $S \subset T$ of all the treatment combinations, how do we define main effects and interactions in the fraction? Here we adopt the approach of Rao [15] (see [3]): We restrict the contrast functions of the full factorial experiment to the subset $S$. Thus we let $\hat{u}$ be the restriction of $u$ to $S$, and let

$$\hat{U}_I = \{\hat{u} : u \in U_I\}. \tag{4}$$

$\hat{U}_I$ denotes the set of restrictions of all the functions in $U_I$ to the fraction $S$. Since addition and scalar multiplication are defined pointwise, $\hat{U}_I$ is also a subspace (of $\mathbb{R}^S$).

Rao [15, page 129] referred to subsets $S \subset T$ as arrays. The adjective orthogonal came a bit later, as did the representation of arrays as matrices. The matrix representation allows elements of the array to be repeated. An array with no repeated elements – a subset of $T$ – is called a simple array. We will also use the term fraction for such subsets. In this paper, we only deal with simple arrays.

In any event, Rao’s crucial discovery was the parameter known as strength.

**Definition 2.2.** $S$ has strength $t \geq 1$ if, for every $I = \{i_1, \ldots, i_t\} \subset \{1, \ldots, k\}$, the projection of $S$ onto the factors $i_1, \ldots, i_t$ consists of $\lambda_I$ copies of the full factorial $A_{i_1} \times \cdots \times A_{i_t}$.

Note that for a symmetric array $S$, the multiplicities $\lambda_I$ are all equal to a common value $\lambda$, the index of the array. As is well known, it follows from the definition that if $S$ has strength $t$ then it also has strength $t'$ for all $t' \lt t$. A convenient equivalent definition of strength is the following [2, Corollary 5.2].
Lemma 2.3. The fraction $S$ has strength $t$ iff

$$S \perp \bigvee_{i \in I} A_i$$

for every $I \subset \{1, \ldots, k\}$ of size $t$.

While Rao did not define aliasing or resolution, he came very close to doing so. The definition of aliasing that follows allows us to define resolution in exactly the same way as Box and Hunter do in regular fractions [6, page 319].

Definition 2.4. $U_I$ and $U_J$ are completely aliased in the fraction $S$ if $\hat{U}_I = \hat{U}_J$, unaliased in $S$ if $\hat{U}_I \perp \hat{U}_J$, and partially aliased in $S$ otherwise.

$S$ has resolution $R$ if, for each $p$, every $p$-factor effect is unaliased with every effect having fewer than $R - p$ factors.

It is straightforward to see that a fraction having resolution $R$ also has resolution $R'$ for all $R' < R$.

We quote the following theorem and corollary from [3, Theorem 3.4(a) and Corollary 3.5]. We include the brief proof of the corollary for convenience.

Theorem 2.5. Let $S$ be a fraction of strength $t$. Let $I, J \subset \{1, \ldots, n\}$ with $|I \cup J| \leq t$. If $I \neq J$, then $\hat{U}_I \perp \hat{U}_J$.

Corollary 2.6. If $S$ has strength $t$ then it has resolution $t + 1$.

Proof. Suppose $S$ has strength $t$, and let $I$ and $J$ be subsets of $\{1, \ldots, k\}$ such that

$$|I| = p \text{ and } |J| \leq t - p.$$  

By Theorem 2.5, $\hat{U}_I \perp \hat{U}_J$. Thus no interaction of $p$ factors is aliased with any interaction of at most $t - p$ factors. But this means that $S$ has resolution $t + 1$.  

Corollary 2.6 implies that if $S$ has maximum strength $t$ then $S$ has resolution $R \geq t + 1$. We now show that $R$ cannot exceed $t + 1$.

Theorem 2.7. If a fraction $S$ has maximum strength $t$, then $S$ has maximum resolution $t + 1$.

Proof. To show that $S$ does not have resolution $t + 2$, we must produce $I, J \subset \{1, \ldots, k\}$ such that $|J| < t + 2 - |I|$ but $\hat{U}_I \not\perp \hat{U}_J$.

Since $S$ does not have strength $t + 1$, there exists a set $K \subset \{1, \ldots, k\}$ such that $|K| = t + 1$ and $S \not\perp \mathcal{C}_K$, where $\mathcal{C}_K = \bigvee_{i \in K} A_i$. That means there exists a block $B \in \mathcal{C}_K$ such that $S \not\perp B$.

Now $|K| \geq 2$, so we may write $K = I \cup J$, where both $I$ and $J$ are nontrivial and $I \cap J = \emptyset$. Since $K = I \cup J$, we have $\mathcal{C}_K = \mathcal{C}_I \vee \mathcal{C}_J$ by Lemma 2.1, so there exist $B' \in \mathcal{C}_I$ and $B'' \in \mathcal{C}_J$ such that $B = B' \cap B''$.

Let $u = 1_{B'} - \pi(B')1$ and $v = 1_{B''} - \pi(B'')1$. Then $u \in U_{\mathcal{C}_I}$ and $v \in U_{\mathcal{C}_J}$. Using equation (3) we have the orthogonal sums

$$u = \sum_{I' \subseteq I} u_{I'}, \quad v = \sum_{J' \subseteq J} v_{J'}.$$
where \( u_I \in U_I \) and \( v_J \in U_J \). Now if \( I' \subset I \) and \( J' \subset J \), then \( I' \neq J' \) (in fact they are disjoint); moreover, if \( I' \neq I \) or \( J' \neq J \), then \( |I' \cup J'| \leq t \), and thus \( (\hat{u}_I, \hat{v}_J) = 0 \) by Theorem 2.5. Hence \((\hat{u}, \hat{v}) = (\hat{u}_I, \hat{v}_J)\). We will show that \((\hat{u}, \hat{v}) \neq 0\). Then \((\hat{u}_I, \hat{v}_J) \neq 0\), and thus \(\hat{U}_I \sim \hat{U}_J\). Now

\[
(\hat{u}, \hat{v}) = \sum_{s \in S} u(s)v(s)
= \sum_{s \in S} (1_B'(s) - \pi(B')1)(1_B''(s) - \pi(B''))1
= |B' \cap B'' \cap S| - \pi(B')|B'' \cap S| - \pi(B'')|B' \cap S| + |S| \pi(B') \pi(B'')
= |T| [\pi(B' \cap B'' \cap S) - \pi(B') \pi(B'' \cap S) - \pi(B'') \pi(B' \cap S) + \pi(S) \pi(B') \pi(B'')].
\]

Since \( S \) has strength \( t \), it is independent of both \( \mathcal{E}_I \) and \( \mathcal{E}_J \) (Lemma 2.3), and so we have \( \pi(B' \cap S) = \pi(B') \pi(S) \) and \( \pi(B'' \cap S) = \pi(B'') \pi(S) \). Moreover, since \( I \cap J = \emptyset \), we have \( \mathcal{E}_I \perp \mathcal{E}_J \) by Lemma 2.1, and thus \( \pi(B' \cap B'') = \pi(B') \pi(B'') \). Therefore,

\[
(\hat{u}, \hat{v}) = |T| [\pi(B' \cap B'' \cap S) - \pi(B') \pi(B'' \pi(S))]
= |T| [\pi(B \cap S) - \pi(B') \pi(B'') \pi(S)]
= |T| [\pi(B \cap S) - \pi(B' \cap B'') \pi(S)]
\neq 0,
\]

since \( S \not\subset B \).

In the following examples, the notation \( OA(N, k, s, t) \) denotes a symmetric orthogonal array of size \( N \) (the number of “runs”), \( k \) factors, \( s \) symbols and strength \( t \). We represent such an array as a \( k \times N \) matrix.

**Example 2.8.** The columns of the \( OA(25, 3, 5, 2) \)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4
1 & 0 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0
0 & 1 & 2 & 3 & 4 & 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 & 1 & 2 & 3 & 4 & 1 & 3 & 4 & 2 & 0 & 1 & 2
\end{bmatrix}
\]

represent a 1/5 fraction of a \( 5^3 \) factorial design, and is nonregular [11]. Since it has strength 2, we know that none of the three main effects are aliased with each other, but two-factor interactions are aliased with main effects.

**Example 2.9.** Consider the following simple orthogonal array of form \( OA(54, 4, 3, 3) \):

\[
\begin{bmatrix}
\]

Its columns constitute 2/3 of a \( 3^4 \) factorial, which cannot be a regular fraction. Since it has strength 3, main effects are not aliased with each other or with two-factor interactions, but may be aliased with three-factor interactions. The four-factor interaction will not be a contrast in the fraction.
Example 2.10. The solution set of the equation \( X_1 + X_2 + X_3 + 2X_4 \equiv 0 \pmod{4} \) forms a 1/4 fraction of a 4\(^4\) factorial, and is an \( OA(64, 4, 4, 2) \). It does not have strength 3, as its projection on the first three factors is not a complete 4\(^3\) factorial, but is rather the juxtaposition of 2 copies each of the fractions of a 4\(^3\) factorial given by \( X_1 + X_2 + X_3 \equiv 0 \pmod{4} \) and \( X_1 + X_2 + X_3 \equiv 2 \pmod{4} \). Thus it has (maximum) resolution 3: main effects are unaliased with each other, but some two-factor interactions are aliased with main effects.

We can even say a bit more about where the aliasing is occurring. Let us call the factors \( A, B, C, \) and \( D \). If we project the fraction on any set of three factors that includes \( D \), we indeed get a complete 4\(^3\) factorial design. Thus those three main effects and all their interactions will be unaliased in the fraction. Aliasing between main effects and two-factor interactions is limited to the three factors other than \( D \).

One might be tempted to view this as a regular fraction with defining contrast \( ABCD^2 \), based on the defining equation, and to conclude that its resolution should be 4 since the “wordlength” of \( ABCD^2 \) is 4. However, the fraction is a solution set of an equation over \( \mathbb{Z}/4 \), not \( GF(4) \). According to our definition, this fraction is not regular and the usual wordlength algorithm need not apply – indeed it doesn’t.

Example 2.11. Consider the “compound” orthogonal array \( O \) formed from the arrays

\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\
0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
\end{bmatrix}
\]

as follows: repeat each column of \( B \) four times, and stack the columns of \( A \) over each group of four [8, page 147]. We thus have a mixed-level orthogonal array with \( k = 7 \) factors and size \( N = 36 \), whose columns constitute a 1/18 fraction of a 2\(^3\)3\(^4\) experiment. In general, if \( A \) and \( B \) have (maximum) strengths \( t_1 \) and \( t_2 \), then the composite array has (maximum) strength \( \min(t_1, t_2) \) [12, page 234]. In this example, \( t_1 = t_2 = 2 \) so \( O \) has strength 2 and thus Box-Hunter resolution 3.

As with Example 2.10, knowing the method of construction allows us to say more about the aliasing structure of the fraction. It turns out that the projection of this fraction on any “mixed” selection of four factors such that two come from \( A \) and two from \( B \) will constitute a full 2\(^2\)3\(^2\) factorial, and so any “mixed” projection on 2 or 3 factors will be an array of full strength. To illustrate what we can deduce, let us write \( F_1, F_2, \) and \( F_3 \) for the factors of \( A \) and \( G_1, \ldots, G_4 \) for those of \( B \) (as in [8]).

- The main effects of \( F_1, F_2, G_1 \) and \( G_2 \) together with all their interactions are unaliased in the fraction, and similarly for other choices of two \( F \)’s and two \( G \)’s.
- Any two interactions \( F_i \times F_j \) and \( G_k \times G_\ell \) are unaliased in the fraction since they involve four factors, two from \( A \) and two from \( B \). The same applies to any main effect \( F_i \) and any three-factor interaction \( F_j \times G_k \times G_\ell \), for example.

Note that similar reasoning cannot be applied to \( F_1 \) and \( F_2 \times F_3 \times G_1 \), as the projection of the fraction on these four factors only has strength 2. Indeed, by examining contrasts it can be seen that \( F_1 \) and \( F_2 \times F_3 \times G_1 \) are partially aliased.
• Any mixed two-factor interactions $F_i \times G_k$ and $F_j \times G_\ell$ are unaliased in the fraction, since they involve (at most) four factors, two from each of $A$ and $B$ (fewer if $i = j$ or $k = \ell$). Moreover, every mixed two-factor interaction is unaliased with every main effect. This is the essence of Theorem 1 of [8]).

• The array $A$ is a regular $2^{3-1}$ fraction with defining relation $F_1 F_2 F_3 = I$, and it is not hard to see that its aliasing patterns (similar to those of our example in Section 1) also hold in the compound array. Similarly, we may verify that $B$ is a regular $3^{4-2}$ fraction with defining relations $I = G_1 G_2 G_3 = G_2 G_2^2 G_4$, from which we may deduce the aliasing patterns among these four factors in the compound array.

It will be noted from these examples that we can more readily show what is not aliased than what is. That is the essence of Box-Hunter resolution. In regular fractions we can avail ourselves of the standard algorithm to show what is aliased, and there is no partial aliasing. With nonregular fractions there is no general algorithm, and aliasing will depend on the method by which the fraction is constructed.

**Remark 2.12.** We have allowed fractions to have strength 1 and resolution 2. These values have not often been considered in practice in the past, as a fraction must have strength 2 (resolution 3) in order for main effects to be unaliased in it. Strength 1 guarantees merely that main effects contrasts remain contrasts in the fraction. Arrays of strength 1 are the subject of current statistical research, where they are called *group screening designs* in the regular case and *supersaturated designs* in general.

It is even possible to allow $t = 0$ in our definition of strength (take an empty join to be the trivial partition $\{T\}$). Then a fraction has maximum strength 0 if it does not even have strength 1. Similarly, all fractions vacuously have resolution 1, and with a bit of elaboration one may show that all ensuing results extend to this case. As there is no practical need for this, we have omitted it.

### 3 Other approaches to resolution and aliasing

Box-Hunter resolution is universally used in assessing the quantity of information we may extract from regular fractions. In Section 2 we reviewed the extension of this definition to non-regular designs, and proved the key property that the (maximum) resolution equals the (maximum) strength plus 1.

We close with a review of alternative approaches to resolution and aliasing, particularly those that grow out of considerations of estimation.

Several authors, such as P. W. M. John [13, page 152] and Raktoe, Hedayat and Federer [14, page 88], define a fraction to have resolution $R$ if all contrasts belonging to effects of order at most $\lceil R/2 \rceil - 1$ are estimable. (An effect is of order $k$ if it involves exactly $k$ factors. The symbol $\lceil x \rceil$ denotes the greatest integer not exceeding $x$.) This is accompanied by the assumption that certain high-order effects are absent, an assumption not present in Box and Hunter’s original formulation (see [3] for a discussion of this issue). One consequence of their definition is that while a design of strength $t$ has resolution $t + 1$, it may, for example, have resolution $R$ but maximum strength $R - 2$ [14, page 174]. Hedayat, Sloane and Stufken [12, page 281] add a further assumption to deal with this problem.
Dey and Mukerjee [9, page 18] define a fractional design to have resolution \((f, t)\), \(f \leq t\), if contrasts belonging to effects of order at most \(f\) are estimable when effects of order greater than \(t\) are absent. They show that fractions of strength \(g\) are universally optimal as long as \(f + t = g\), and then define the resolution of such a fraction to be \(g + 1\) [9, Theorem 2.6.1 and Remark 2.6.2].

Box-Hunter resolution distinguishes itself from these approaches in a couple of ways:

- It is a combinatorial property of the underlying fraction, independent of any modeling assumptions (such as the absence of high-order interactions).
- It is a measure of the amount of aliasing in a fraction rather than of the estimability of terms in a model.

We are relying here on the concept of aliasing used in regular fractions (Definition 2.4). We note, however, that aliasing is sometimes defined in terms of bias, specifically as a measure of the biases caused by misspecification of a model (cf. [7, page 7] and [14, page 95]). The relation between the two views of aliasing is discussed in [3].

**A Appendix**

In this section we complete the connection between resolution, strength and wordlength in regular fractional factorials.

Throughout the section we adopt the following notation and definitions. \(I^c\) denotes the complement of \(I\) (in this case, with respect to \(\{1, \ldots, k\}\)). Column vectors will be denoted by boldface lower case letters and the transpose of a vector or matrix by \(\prime\). The projection of the vector \(a\) onto the components indexed by \(I\) will be denoted \(a_I\).

Note that the span of a set of vectors always includes the zero vector. Since we will need to refer to the set of all nontrivial linear combinations of a set of vectors, we define \(\text{span}^*\{a_1, \ldots, a_n\} = \{\sum_{i=1}^n c_i a_i : c_i \neq 0 \text{ for some } i\}\). Note that if \(A\) is a set of vectors, then

\[
\text{span}^*(A) = \begin{cases} 
\text{span}(A) & \text{if } A \text{ is linearly dependent,} \\
\text{span}(A) \setminus \{0\} & \text{if } A \text{ is linearly independent.}
\end{cases}
\]

Here \(0\) denotes the zero vector. Finally, the \((\text{Hamming})\) weight of a vector \(a\), denoted \(\text{wt}(a)\), is the number of nonzero components of \(a\).

Throughout this section the set of treatment combinations is taken to be \(T = (GF(s))^k\). Recall that a fraction \(S \subset T\) is regular if it is the solution set to a system of linear equations over \(GF(s)\). The following lemma will be useful:

**Lemma A.1.** Let \(S\) be a regular fraction given by the solution set to

\[
\begin{aligned}
\mathbf{a}_1' \mathbf{x} &= a_1 \\
\vdots \\
\mathbf{a}_m' \mathbf{x} &= a_m,
\end{aligned}
\]

where \(\mathbf{a}_1, \ldots, \mathbf{a}_m \in (GF(s))^k\) are independent and \(a_i \in GF(s)\). Then \(S\) has strength \(t\) if and only if the projections of the coefficient vectors \(\mathbf{a}_1, \ldots, \mathbf{a}_m\) onto any \(k - t\) factors are linearly independent.
Proof. Since (5) is a system with $m$ independent equations in $k$ variables, $|S| = s^{k-m}$. Thus $S$ has strength $t$ if and only if for any choice of $t$ factors, the projection of $S$ onto these factors contains exactly $\lambda = s^{k-t-m}$ copies of $u$ for every $u \in (GF(s))^t$.

Choose a subset $I \subset \{1, \ldots, k\}$ with $|I| = t$. Without loss of generality, let $I = \{1, \ldots, t\}$. For each $j \in I$, fix $u_j \in GF(s)$, and consider the system of equations obtained by substituting $x_j = u_j$ into (5) for $j \leq t$. This system is given by

$$
\begin{align*}
\{ & a'_{1,1c_1}x_{1c_1} = b_1 \\
& \quad \vdots \\
& a'_{m,mc_m}x_{mc_m} = b_m,
\end{align*}
$$

(6)

where $b_i = a_i - \sum_{j=1}^t a_{ij}u_j$.

Now assume that $S$ has strength $t$. Then the projection onto the $t$ factors indexed by $I$ contains every element of $(GF(s))^t$ exactly $s^{k-t-m}$ times. That is, for every $u = (u_1, \ldots, u_t)' \in (GF(s))^t$, $S$ contains exactly $s^{k-t-m}$ elements of the form $[v]$, where $v \in (GF(s))^{k-t}$. But these $v$ are exactly the solutions to (6). Therefore (6) has exactly $s^{k-t-m}$ solutions. Since (6) is a system in $k - t$ variables with $m$ equations, this implies that the coefficient vectors $a'_{1,1c_1}, \ldots, a'_{n,mc_m}$ are independent. Thus if $S$ has strength $t$, then the projections of the coefficient vectors $a_i$ onto any $k - t$ factors are linearly independent.

Conversely, if the $a'_{i,1c_1}$ are linearly independent for any choice of $I$ with $|I| = t$, then all systems of equations in $k - t$ variables generated by fixing any $t$ of $x_1, \ldots, x_k$ have exactly $s^{k-t-m}$ solutions, which implies that $S$ has strength $t$. □

Theorem A.2. Let $S$ be a regular fraction given by the solution set to

$$
\begin{align*}
\{ & a'_1x = a_1 \\
& \quad \vdots \\
& a'_mx = a_m,
\end{align*}
$$

(7)

where $a_1, \ldots, a_m \in (GF(s))^k$ are independent and $a_i \in GF(s)$. Let $t$ be the maximum strength of $S$ and $w$ the minimum weight of the vectors in span$(a_1, \ldots, a_m)$. Then $t = w - 1$.

Proof. Let $A = \{a_1, \ldots, a_m\}$. Note that if $b$ is any vector in span$(A)$ that is not itself in $A$, then there exists a set $B \subseteq (GF(s))^k$ of linearly independent vectors such that $B = \{b, b_2, \ldots, b_m\}$ and span$(B) = \text{span}(A)$. Further, there exist $b_1, \ldots, b_m \in GF(s)$ such that $S$ is the solution set to the system

$$
\begin{align*}
b'x &= b_1 \\
b'_2x &= b_2 \\
& \quad \vdots \\
b'_mx &= b_m.
\end{align*}
$$

Thus we can assume without loss of generality that the weight of one of the coefficient vectors in (7) is equal to $w$.

Assume that $\text{wt}(a_1) = w$, say. Rearranging the order of the variables if necessary, we can assume $a_1 = (a_{11}, \ldots, a_{1w}, 0, \ldots, 0)'$ with $a_{1j} \neq 0$ for $j = 1, \ldots, w$. 

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Fix \( \mathbf{u} = (u_1, \ldots, u_{w-1}) \) and let \( Z_{\mathbf{u}} = \{ \mathbf{z} : \mathbf{z} = [\mathbf{u}] \} \). Now, every element in \( Z_{\mathbf{u}} \) is a solution to (7). Substituting \( x_i = u_i \) for \( i = 1, \ldots, w-1 \), we see that for every \( [\mathbf{u}] \in Z_{\mathbf{v}} \), \( \mathbf{v} \) is a solution to
\[
\begin{align*}
  a_1wv_w &= c_1 \\
  a_2wv_w + \cdots + a_kv_k &= c_2 \\
  \vdots \\
  a_mwv_w + \cdots + a_kv_k &= c_m,
\end{align*}
\]
where \( c_i = a_i - \sum_{j=1}^{w-1} a_{ij}u_j \) for \( i = 1, \ldots, m \). We now have a system of \( m \) equations in \( k-(w-1) \) variables, which has at least \( s^{k-m-(w-1)} \) solutions. Thus every element \( \mathbf{u} \) of \( (GF(s))^{w-1} \) occurs at least \( s^{k-m-(w-1)} \) times in the projection of \( S \) onto the first \( t \) factors. Since (7) is a system with \( m \) independent equations in \( k \) variables, \( |S| = s^{k-m} \), and therefore \( S \) has at least strength \( w-1 \), that is, \( t \geq w-1 \).

Since \( \text{wt}(\mathbf{a}_1) = w \), the coefficient vector \( \mathbf{a}_1 \) has \( k-w \) zero components. By Lemma A.1, no projection of \( A \) onto \( k-t \) components can contain the zero vector, so we must have \( k-t > k-w \), and therefore \( t \leq w-1 \) (as \( t \) is an integer). Thus \( t = w-1 \).

Theorem 4.6 of [12] is a coding-theoretic version of Theorem A.2 (technically, the homogeneous case). Theorem 8.2.2 of [9] is the statement \( t \geq w-1 \) (the authors define resolution to be the minimum weight of a defining pencil).

As usual, the connection with group theory is made by defining a finite commutative group \( G \) having as elements (words) the symbols \( A_1^{a_1} \cdots A_k^{a_k} \), where \( a_i \in GF(s) \). Multiplication is defined using the rules of exponents, but where the addition and multiplication of exponents is governed by the operations in \( GF(s) \). The correspondence
\[
(a_1, \ldots, a_k) \leftrightarrow A_1^{a_1} \cdots A_k^{a_k}
\]
converts the addition and scalar multiplication of \( (GF(s))^k \) to multiplication and exponentiation in \( G \), and a subspace of \( (GF(s))^k \) spanned by the vectors \( \mathbf{a}_1, \ldots, \mathbf{a}_m \) corresponds to the subgroup generated by the corresponding words of \( G \). (A technical point: Exponentiation by elements of \( GF(s) \) that are not integers is not an ordinary group operation. Such exponents must really be viewed as operators on \( G \). We are requiring that subgroups be closed under the action of these operators.) Linear independence corresponds to the obvious way to independence of elements of \( G \). Last, the weight of a vector \( \mathbf{a} \) equals the length of the corresponding word.

Thus a regular fraction is defined by a set of \( m \) words, or by the subgroup that they generate (the defining subgroup of the fraction). We may now rewrite Theorem A.2 as follows:

**Corollary A.3.** Let \( S \) be a regular fraction with defining subgroup \( H \). Let \( t \) be the maximum strength of \( S \) and \( w \) the minimum length of the non-identity words in \( H \). Then \( t = w-1 \).

Finally, by combining Corollary A.3 with Theorem 2.7 we complete the connection between resolution and wordlength:

**Corollary A.4.** Let \( S \) be a regular fraction with defining subgroup \( H \). Let \( R \) be the maximum resolution of \( S \) and \( w \) the minimum length of the non-identity words in \( H \). Then \( R = w \).
References


