The decomposability of simple orthogonal arrays on 3 symbols having $t + 1$ rows and strength $t$

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Abstract

It is well-known that all orthogonal arrays of the form $OA(N, t+1, 2, t)$ are decomposable into $\lambda$ orthogonal arrays of strength $t$ and index 1. While the same is not generally true when $s = 3$, we will show that all simple orthogonal arrays of the form $OA(N, t + 1, 3, t)$ are also decomposable into orthogonal arrays of strength $t$ and index 1.

Key words. Orthogonal array, decomposable orthogonal array, simple orthogonal array, regular orthogonal array.

1 Introduction

In this paper, we continue an ongoing discussion about the decomposability of orthogonal arrays having $t + 1$ factors. It is obvious that the juxtaposition of several such orthogonal arrays again yields an orthogonal array of the same strength. We will consider the reverse problem: When can an orthogonal array be partitioned into smaller arrays of the same strength?

When $s = 2$, Seiden and Zemach [7] have shown that every orthogonal array of the form $OA(N, t + 1, 2, t)$ is the juxtaposition of orthogonal arrays having index 1. This eventually led to a classification of these orthogonal arrays into three different types provided by Cheng [1].

Unfortunately, this result does not extend beyond the binary case. When $s = 3$, there exist orthogonal arrays that are not decomposable, as we will see in Section 5. How to identify whether a given orthogonal array is decomposable has been investigated only for a small number of cases (see [4]). Early results suggest that the number of repeated columns plays an important role in this identification. The goal of this paper is to prove that the assumption of simplicity assures the decomposability of orthogonal arrays of the form $OA(N, t + 1, 3, t)$ (see Corollary 5.7).

We will denote the field with $s$ elements by $\mathbb{Z}_s$ when $s$ is a prime, and by $GF(s)$ when $s$ is a prime power. The cardinality of a set $S$ will be denoted by $|S|$. Column vectors will
be denoted by boldface letters and transposition by \( ' \). We will use the usual set notation to denote multisets, for example, \( A = \{a, a, b, b, c\} \) and \( B = \{b, b, c, c\} \) are multisets on the set \( S = \{a, b, c\} \). We will use + for multiset sums: \( A + B = \{a, a, b, b, b, b, c, c, c\} \). If a multiset contains every element of a set \( S \) the same number of times \( n \), we will denote that multiset by \( n \ast S \). For example, \( 2 \ast S = \{a, a, b, b, c, c\} \).

2 Decomposability and regularity

Definition 2.1 Let \( S \) be a set of \( s \) elements. An orthogonal array of size \( N \), \( k \) constraints (or factors), \( s \) levels and strength \( t \), denoted \( OA(N, k, s, t) \), is a multiset of \( N \) elements from \( S^k \) such that the projection of \( O \) onto any subset of \( t \) factors contains every \( t \)-tuple in \( S^t \) the same number of times. The common frequency is denoted by \( \lambda \) and is called the index of the orthogonal array. We will call an orthogonal array simple if it has no repeated columns (cf. [6]).

It is easy to see that \( \lambda = N/s^t \), and that arrays of index 1 are the smallest conceivable.

In this paper, we will investigate when an orthogonal array can be partitioned into smaller orthogonal arrays of the same strength.

It is clear that if \( O_i \) is an orthogonal array of the form \( OA(N_i, k, s, t) \) of index \( \lambda_i \) for \( i = 1, ..., n \), and \( O = O_1 + \cdots + O_n \), then \( O \) is also an orthogonal array of strength \( t \) and has index \( \lambda = \sum_{i=1}^{n} \lambda_i \).

Definition 2.2 Let \( O \) be an orthogonal array of strength \( t \). We say \( O \) is decomposable if there exist orthogonal arrays \( O_1, O_2 \subset O \) of strength \( t \) such that we have \( O = O_1 + O_2 \). We will call \( O_1 \) and \( O_2 \) components of \( O \).

We say that \( O \) is completely decomposable if

\[
O = O_1 + \cdots + O_n
\]

(1)

where each of the components \( O_i, i = 1, ..., n \), has strength \( t \) and index 1. The expression in (1) is called a complete decomposition of \( O \).

Note that any complete decomposition (1) of an orthogonal array \( O \) of index \( \lambda \) contains \( n = \lambda \) components.

When \( s \) is a power of a prime, we can choose \( S \) to be \( GF(s) \), the field with \( s \) elements.

Definition 2.3 Let \( s \) be a prime power. A subset \( A \) of \( GF(s)^k \) is an affine space if \( A = L + v \), where \( L \) is a subspace of the vector space \( GF(s)^k \) and \( v \in GF(s)^k \).

Note that an affine space of \( G(s)^k \) is the solution set of a set of linear equations in \( k \) variables over the field \( GF(s) \), and vice versa.

We define the following important property for orthogonal arrays:

Definition 2.4 Let \( s \) be a power of a prime. A simple orthogonal array of the form \( OA(N, k, s, t) \) is called regular if its columns form an affine space of \( GF(s)^k \).

Regularity will play an important role in the proof of our main result.
3 Preliminary results

Definition 3.1 Let $S$ be a set and $n$ a positive integer. The Hamming distance between $x, y \in S^n$ is defined to be $d(x, y) = |\{i : x_i \neq y_i\}|$. We define $F_n(S)$ to be the set of all functions $f : S^n \rightarrow S$ that satisfy the following condition:

\[ d(x, y) = 1 \implies f(x) \neq f(y). \]

The families $F_n(S)$ are closely related to orthogonal arrays of the form $OA(s^n, n+1, s, n)$. We have

Lemma 3.2 Let $f : S^n \rightarrow S$. Then the set \{(x, f(x))' : x \in S^n\} forms an orthogonal array of the form $OA(s^n, n+1, s, n)$ if and only if $f \in F_n(S)$.

Theorem 3.3 Let $s = 2$ or $3$. Then the set $F_n(S)$ consists precisely of the functions $f : S^n \rightarrow S$ of the form

\[ f(x_1, \ldots, x_n) = c + \sum_{i=1}^{n} a_i x_i, \]

where $c, a_i \in \mathbb{Z}_s$ and $a_i \neq 0$.

Proof. The case for $s = 2$ is a direct consequence of the fact that every function $f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$ can be expressed as a polynomial in $n$ variables, where no exponent is larger than 1 (cf. [2], [8]).

Let $s = 3$. Without loss of generality, assume that $f(0, \ldots, 0) = 0$. Let $a_i = f(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is the $i^{th}$ entry. Since $f \in F_n(S)$, we have $a_i \neq 0 \forall i$.

Claim: $f(x) = \sum_{i=1}^{n} a_i x_i$ for all $x \in S^n$.

Proof of claim. We will use induction on the weight $w$ of $x$, where $w$ is the number of nonzero components of $x$.

It is easily verified that $f(0, \ldots, 0, x_i, 0, \ldots, 0) = a_i x_i$ for all $i$.

Now assume the claim is true for all $x \in S^n$ with weight less than $w$, where $w \geq 2$.

Let $x$ have weight $w$. To show: $f(x) = \sum_{i=1}^{n} a_i x_i$.

For ease of notation, we will suppress all components of $x$ that are zero and write $x = (x_1, x_2, \ldots, x_w)$. Let $y = \sum_{i=1}^{w-1} a_i x_i$ and $z = \sum_{i=2}^{w} a_i x_i$. Since $f \in F_n(S)$, we have

\[ f(0, x_2, \ldots, x_w) \neq f(x) \text{ and } f(x_1, \ldots, x_{w-1}) \neq f(x), \]

and by induction, $f(x_1, \ldots, x_{w-1}) = y$ and $f(0, x_2, \ldots, x_w) = z$. Thus

\[ f(x) \neq y \text{ and } f(x) \neq z. \]

Now we consider two cases:

(i) $y \neq z$: Since $\sum_{i=1}^{w} a_i x_i$ is different from $y$ and $z$ and $s = 3$, it follows from (2) that $f(x) = \sum_{i=1}^{w} a_i x_i$.

(ii) $y = z$: In this case, $f(x_1, \ldots, x_{w-1}, -x_w) = \sum_{i=1}^{w-1} a_i x_i - a_w x_w$, which is different from $y$ and $f(x)$. Thus (i) implies that $f(x) = \sum_{i=1}^{w} a_i x_i$.

\[ \blacksquare \]

Corollary 3.4 When $s = 2$ or $3$, every orthogonal array of the form $OA(s^t, t+1, s, t)$ is regular.

Hedayat, Stufken and Su [5] provided a proof of Corollary 3.4 for the case when $s = 3$ and $t \geq 2$. 

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Lemma 3.5 \[ |F_n(S)| = 2 \] if \( s = 2 \), and \[ |F_n(S)| = 3(2^n) \] if \( s = 3 \).

In the proof of our main theorem, we will utilize the following:

Lemma 3.6 Let \( n > 1 \), \( S \) a finite set. Then \( f : S^n \to S \) is an element of \( F_n(S) \) if and only if any function derived from \( f \) by fixing \( m \) components of its argument is an element in \( F_{n-m}(S) \).

We will call a function derived from \( f \) by fixing all but the \( k \)th component a \( k \)th section of \( f \).

For the remainder of this paper, we will assume that \( S = \mathbb{Z}_3 = \{0, 1, 2\} \), and denote \( F_n(S) \) by \( F_n \).

It is convenient to introduce the following terminology:

Definition 3.7 Let \( f, g \in F_n \). We will say that \( f \) and \( g \) are parallel if \( f(x) \neq g(x) \forall x \). A maximal set of parallel functions in \( F_n \) will be called a parallel class of \( F_n \).

Lemma 3.8 The parallel class of a function \( f \in F_n \) is precisely the set \( \{f, f - 1, f + 1\} \).

Note that for fixed \( k \), two different \( k \)th sections of \( f \in F_n \) are parallel.

Proposition 3.9 Let \( n \geq 1 \). Let \( f, g, \sigma, \tau \in F_n \) such that \( f \) and \( g \) are parallel and \( \sigma \) and \( \tau \) are parallel. If for all \( x \in S^n \), \( \{f(x), g(x), \sigma(x), \tau(x)\} \supset S \), then \( f, g, \sigma \) and \( \tau \) are in the same parallel class.

Since \( f, g, \sigma, \tau \) all map into \( S \), which is a set with three elements, this means that for each \( x \in S^n \), the set \( \{f(x), g(x), \sigma(x), \tau(x)\} \) is a multiset that contains every element of \( S \).

Proof of Proposition 3.9. Using Theorem 3.3, we can write \( f(x) = A + a'x \), where \( x = (x_1, \ldots, x_n) \), \( a = (a_1, \ldots, a_n) \), \( a_i, A \in S \) and \( a_i \neq 0 \). Similarly, we write \( g(x) = B + a'x \), \( \sigma(x) = C + b'x \), \( \tau(x) = D + b'x \), where \( A \neq B, C \neq D \).

To show: \( a = b \).

By assumption, \( \{A, B, C, D\} \) contains \( S \). Without loss of generality, we can choose notation such that \( A = C = 0, B = 1, D = 2 \).

Now suppose \( a \neq b \). Then choosing \( x \) such that \( (a - b)'x = 2 \) yields a contradiction. Therefore we must have \( a = b \).

The proof of our main theorem will require the use of the following idea:

Let \( S = \{a, b, c\} \). Let \( \alpha = (a, a, b), \beta = (c, b, c) \) be ordered triples. To transform \( \alpha \) and \( \beta \) into triples with three distinct entries each, we carry out an exchange of a component of \( \alpha \) with the corresponding component of \( \beta \). We exchange \( a \) with \( c \) to get the ordered triples \( \pi = (c, a, b) \) and \( \gamma = (a, b, c) \). Note that this is the “fastest” way to achieve our goal. In fact, we always only need at most one exchange in order to construct triples with distinct entries.

In general, we have the following:

Lemma 3.10 Let \( S \) be a set of size 3. Let \( \alpha = (\alpha_0, \alpha_1, \alpha_2), \beta = (\beta_0, \beta_1, \beta_2) \in S^3 \) such that \( \alpha_i \neq \beta_j \), \( i = 0, 1, 2 \) and \( \{\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\} = 2^S \).

Then there exists exactly one element \( \pi = (\pi_0, \pi_1, \pi_2) \in S^3 \) such that

(i) \( \pi_i \neq \pi_j \) whenever \( i \neq j \) (i.e. \( \pi \) is a permutation on \( S \)),

(ii) \( \pi_i = \alpha_i \) or \( \beta_i \), \( i = 0, 1, 2 \), and

(iii) \( \{\pi_0, \pi_1, \pi_2\} \) contains at least two elements of \( \{\alpha_0, \alpha_1, \alpha_2\} \).
If we define 

$$\gamma_i = \begin{cases} \alpha_i & \text{if } \pi_i = \beta_i \\ \beta_i & \text{if } \pi_i = \alpha_i \end{cases}$$

for $i = 0, 1, 2$, then $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ is also a permutation on $S$, and $\pi$ and $\gamma$ are parallel.

The proof of Lemma 3.10 consists of verifying all possible cases (cf. [3]).

**Remark 3.11** In the case when $\alpha$ and $\beta$ are permutations, we have $\pi = \alpha$ and $\gamma = \beta$. For each other form that $\alpha$ and $\beta$ can take, $\pi$ and $\gamma$ are obtained by exchanging an entry of $\alpha$ with one of $\beta$. The exchange takes place in the coordinate where both $\alpha$ and $\beta$ have an entry that occurs twice.

The proof of our main result is based on a scheme for rearranging the columns of an orthogonal array $O$. We will divide $O$ into two sets and transform each into an orthogonal array by interchanging certain columns between the sets. When carrying out exchanges of columns, we will look at sets of 3 columns at a time and only consider their entries in a specified row. These entries will correspond to the triples $\alpha$ and $\beta$ in Lemma 3.10. We will use Remark 3.11 to decide whether an exchange is necessary, and, if it is, which columns to interchange.

### 4 An illustrative example

The proof of our main result is constructive and fairly technical; we will use an example to illustrate its basic principle.

Consider the following simple orthogonal array of form $OA(54, 4, 3, 3)$:

$$\begin{bmatrix} 00000000 & 11111111 & 22222222 & 00000000 & 11111111 & 22222222 \\ 00011122 & 00111122 & 00111122 & 00011122 & 00011122 & 00011122 \\ 01201201 & 01201201 & 01201201 & 01201201 & 01201201 & 01201201 \\ 12221120 & 21010202 & 10202121 & 00102011 & 02121010 & 21010202 \end{bmatrix}$$

The arrangement of the columns into two blocks that are “lexicographically ordered” is no accident; it will be useful when demonstrating how to decompose the array. In fact, a similar ordering can be accomplished for any orthogonal array and will be utilized in the general argument in Section 5.

We will show that this array is decomposable into two index 1 arrays, that is, we will show that we can write $O$ as the sum of two orthogonal arrays of the form $OA(27, 4, 3, 3)$.

First, we can write $O = A + B$, where

$$A = \begin{bmatrix} 00000000 & 11111111 & 22222222 \\ 00011122 & 00011122 & 00011122 \\ 01201201 & 01201201 & 01201201 \\ 12221120 & 21010202 & 10202121 \end{bmatrix},$$

$$B = \begin{bmatrix} 00000000 & 11111111 & 22222222 \\ 00011122 & 00011122 & 00011122 \\ 01201201 & 01201201 & 01201201 \\ 01020112 & 02121010 & 21010201 \end{bmatrix}$$

Note that $A$ and $B$ are not orthogonal arrays of strength 3 and index 1. For example, the subarray obtained from $A$ by deleting the first row contains the column $(0, 0, 1)'$ twice.

We will exchange columns between $A$ and $B$ to transform both into arrays of strength 3.
Notice that while $A$ and $B$ are not orthogonal arrays, they both satisfy the following: The projection onto the first three rows consists of a full copy of $\mathbb{Z}_3^3$. By Lemma 3.2, both arrays can therefore be described by functions on $\mathbb{Z}_3^3$:

$$A = \left\{ \begin{pmatrix} y \\ f(y) \end{pmatrix} : y \in \mathbb{Z}_3^3 \right\}, 
B = \left\{ \begin{pmatrix} y \\ g(y) \end{pmatrix} : y \in \mathbb{Z}_3^3 \right\}.$$ 

Let us define functions $\sigma_i, \tau_i : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3, i = 0, 1, 2$ as follows:

$$\sigma_i(x) = f(i, x), \quad \tau_i(x) = g(i, x).$$

Let

$$A_i = \left\{ \begin{pmatrix} i \\ \sigma_i(x) \end{pmatrix} : x \in \mathbb{Z}_3 \times \mathbb{Z}_3 \right\}, 
B_i = \left\{ \begin{pmatrix} i \\ \tau_i(x) \end{pmatrix} : x \in \mathbb{Z}_3 \times \mathbb{Z}_3 \right\}.$$ 

Then

$$A_0 = \begin{bmatrix} 000000000 \\ 000112222 \\ 012012012 \\ 122211122 \end{bmatrix}, 
A_1 = \begin{bmatrix} 111111111 \\ 000112222 \\ 012012012 \\ 210102021 \end{bmatrix}, 
A_2 = \begin{bmatrix} 222222222 \\ 000112222 \\ 012012012 \\ 102012012 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 000000000 \\ 000112222 \\ 012012012 \\ 001020101 \end{bmatrix}, 
B_1 = \begin{bmatrix} 111111111 \\ 000112222 \\ 012012012 \\ 021210102 \end{bmatrix}, 
B_2 = \begin{bmatrix} 222222222 \\ 000112222 \\ 012012012 \\ 210102021 \end{bmatrix}.$$ 

and $A = A_0 + A_1 + A_2$ and $B = B_0 + B_1 + B_2$.

Now, if $A$ and $B$ were orthogonal arrays of strength 3, they would have to correspond to functions $f, g$ in $F_3$. The functions $\sigma_i$ would then be $i^{th}$ sections of $f$, and the $\tau_i$ would be $i^{th}$ sections of $g$. Thus $\sigma_i$ and $\tau_i$ would have to satisfy the following conditions:

- For each $i, \sigma_i, \tau_i \in F_2$
- $\sigma_i$ and $\sigma_j$ are parallel whenever $i \neq j$.

Note that in our example, the first condition holds for $i = 1, 2$, but not for $i = 0$. We will therefore concentrate on the case $i = 0$, i.e. on the sets $A_0$ and $B_0$.

First, we will exchange columns between $A_0$ and $B_0$ to construct “new” functions $\sigma_0, \tau_0 \in F_2$. We will first consider the one-variable functions $\sigma_0(i, \cdot)$ and $\tau_0(i, \cdot)$, and turn them into permutations (functions in $F_1$) by interchanging the appropriate columns between $A_0$ and $B_0$. We will then consider the functions $\sigma_0(\cdot, i)$ and $\tau_0(\cdot, i)$, and convert them to permutations in the same way (at this point, the functions $\sigma_0$ and $\tau_0$ will be in $F_2$). If necessary, we will then exchange the appropriate columns to ensure that the second condition is met as well.

Consider the functions $\sigma_0(0, \cdot), \tau_0(0, \cdot)$. The elements $\sigma_0(0, 0), \sigma_0(0, 1), \sigma_0(0, 2)$ and $\tau_0(0, 0), \tau_0(0, 1), \tau_0(0, 2)$ are the entries in the last row of the first three columns in $A_0$ and $B_0$, respectively, and correspond to the triples $\alpha = (1, 2, 2)$ and $\beta = (0, 0, 1)$. Following Remark 3.11, we exchange the second column of $A_0$ with the second column of $B_0$.

We proceed in the same fashion for $i = 1, 2$. When $i = 1$, consider columns 4-6 in $A_0$ and $B_0$ and exchange the respective sixth columns. When $i = 2$, consider columns 7-9 and exchange column 8 between $A_0$ and $B_0$. This results in the following “new” sets $A_0$ and $B_0$:

$$A_0 = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}.$$
Note that the redefined functions $\sigma_0(i, \cdot)$ and $\tau_0(i, \cdot)$ are permutations.

Now consider these sets $A_0$ and $B_0$, and the functions $\sigma_0(\cdot, 0), \tau_0(\cdot, 0)$ they define. These correspond to columns 1, 4, and 7, as indicated below.

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
\end{array}
\]

Again, we can accomplish turning $\sigma_0(\cdot, 0)$ and $\tau_0(\cdot, 0)$ into permutations by exchanging only one set of columns. The arrows indicate the columns that are interchanged. This exchange yields “new” sets $A_0$ and $B_0$:

\[
A_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 & 1 & 2 & 0 & 1 \\
\end{bmatrix}
\]

\[
B_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
\end{bmatrix}
\]

Note that as a result of this exchange, $\sigma_0(1, \cdot), \tau_0(1, \cdot)$ are no longer permutations. In other words, we just “undid” part of what we constructed earlier. We will see, however, that this is only a temporary problem. In fact, assuring that later exchanges of columns don’t “destroy” the results of earlier ones will be a major part of the proof in section 5.

Next consider $\sigma_0(\cdot, 1), \tau_0(\cdot, 1)$, corresponding to columns 2, 5, and 8. According to Remark 3.11, we exchange the fifth column in $A_0$ with the fifth column in $B_0$. We get

\[
A_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 & 1 & 2 & 0 & 1 \\
\end{bmatrix}
\]

\[
B_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
0 & 2 & 1 & 2 & 1 & 1 & 0 & 2 \\
\end{bmatrix}
\]

$(\sigma_0(1, \cdot), \tau_0(1, \cdot)$ are still not permutations.)

Next consider $\sigma_0(\cdot, 2), \tau_0(\cdot, 2)$, i.e. columns 3, 6, and 9. In this step, we exchange the respective sixth columns and obtain the following:

\[
A_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 & 1 & 2 & 0 & 1 \\
\end{bmatrix}
\]

\[
B_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
0 & 2 & 1 & 2 & 1 & 1 & 0 & 2 \\
\end{bmatrix}
\]

Notice that after the last exchange has taken place, not only are the functions $\sigma_0(\cdot, 1), \tau_0(\cdot, 1)$ permutations, but the (new) functions $\sigma_0(1, \cdot)$ and $\tau_0(1, \cdot)$ now are, too. In fact, all functions of the form $\sigma_0(i, \cdot), \sigma_0(\cdot, i), \tau_0(i, \cdot)$ and $\tau_0(\cdot, i)$ are permutations for all $i$. This happens in the general case as well, as we will show in Theorem 5.5. At this point, $\sigma_0, \tau_0 \in F_2$.

However, the condition that $\sigma_i$ and $\sigma_j$ are parallel whenever $i \neq j$ is not satisfied. For example, $\sigma_0(0, 0) = \sigma_2(0, 0) = 1$. 

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To remedy this, we interchange the entire set $A_0$ with the set $B_0$ (that is, rename $A_0$ as $B_0$ and vice versa). With the resulting redefinition of $\sigma_0$ and $\tau_0$, $\sigma_i$ and $\sigma_j$ are now parallel.

Letting $A = A_0 + A_1 + A_2$ and $B = B_0 + B_1 + B_2$, we have

$$A = \begin{bmatrix}
00000000 & 11111111 & 22222222 & 22222222 \\
00111222 & 00011122 & 00011122 & 00011122 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012
\end{bmatrix}$$

and

$$B = \begin{bmatrix}
00000000 & 11111111 & 22222222 & 22222222 \\
00111122 & 00011122 & 00011122 & 00011122 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012 \\
012012012 & 012012012 & 012012012 & 012012012
\end{bmatrix}.$$

Both $A$ and $B$ are orthogonal arrays of form $OA(27,4,3,3)$, and $A + B = O$. Thus $O$ is (completely) decomposable.

5 The decomposition of simple orthogonal arrays of the form $OA(N, t + 1, 3, t)$

Not every orthogonal array of the form $OA(N, t + 1, 3, t)$ is decomposable. For example, the array

$$O = \begin{bmatrix}
0000111222000111222 \\
0120120120120120120 \\
0120120120120120120
\end{bmatrix}$$

is an orthogonal array of form $OA(18,3,3,2)$ that is indecomposable (see [4]). Notice that $O$ is not simple, but contains exactly one repeated column.

In this section we prove that when $s = 3$, every simple orthogonal array of the form $OA(N, t + 1, 3, t)$ is completely decomposable into regular orthogonal arrays.

When $\lambda = 1$, simplicity is guaranteed (and decomposability is no question, as the array is of smallest possible size). When $\lambda = 3$, a simple orthogonal array of the form $OA(\lambda(3^t), t + 1, 3, t)$ is the set $S^{t+1}$, and thus completely decomposable as

$$S^{t+1} = \{ x \in S^{t+1} : c^T x = 0 \} + \{ x \in S^{t+1} : c^T x = 1 \} + \{ x \in S^{t+1} : c^T x = 2 \},$$

where $c \in S^{t+1}$ has all nonzero components. When $\lambda > 3$, simplicity is impossible, as $S^t$ is the largest simple orthogonal array of the form $OA(N, t + 1, 3, t)$. The only interesting case is therefore that of simple orthogonal arrays of the form $OA(2(3^t), t + 1, 3, t)$.

Let us first introduce some notation. Let $O$ be an orthogonal array of the form $OA(2(3^t), t + 1, 3, t)$. We can partition $O$ into 6 sets of $3^{t-1}$ columns as follows:

The first $t$ rows of the array constitute $2^t$ copies of $S^t$ (since $O$ has index 2). This allows us to divide the columns of $O$ into 2 blocks, say $A$ and $B$, such that the first $t$ rows of each block form a full copy of $S^t$.

Note that this partition of $O$ into the matrices $A$ and $B$ is not unique. Further, $A$ and $B$ need not be orthogonal arrays of strength $t$. The partition of $O$ as $O = A + B$ therefore does not imply that $O$ is decomposable.

We can further partition $A$ as $A = A_0 + A_1 + A_2$, where $A_i$ is the set of all columns of $A$ whose first component is $i$. Similarly, we can partition $B$ as $B = B_0 + B_1 + B_2$.

Since the first $t$ rows of $A$ constitute one copy of $S^t$, rows 2 through $t$ of $A_i$ make up one full copy of $S^{t-1}$. The same holds for $B_i$. Therefore we can write

$$A_i = \left\{ \left( \frac{x}{\sigma_i(x)} \right) : x \in S^{t-1} \right\} \quad \text{and} \quad B_i = \left\{ \left( \frac{x}{\tau_i(x)} \right) : x \in S^{t-1} \right\}$$

for $i = 0, 1, 2$, where $\sigma_i, \tau_i$ are functions.
Definition 5.1 We will call the set \( \{\sigma_i, \tau_i : i = 0, 1, 2\} \) a representation of \( O \).

Every orthogonal array of the form \( OA(\lambda p^t, t + 1, p, t) \) can be partitioned into \( \lambda p \) sets in this fashion, regardless of the value of \( p \) (see [3], Section 5.2).

Remark 5.2 Note that every \( OA(2(3^t), t+1, 2, t) \) has at least one representation. Further, distinct orthogonal arrays differ in at least one column. Since the columns determine the functions in the representation, distinct arrays have distinct representations.

The representations of any simple \( OA(2(3^t), t+1, 3, t) \) are characterized as follows:

**Lemma 5.3** If \( O \) is a simple orthogonal array of the form \( OA(2(3^t), t+1, 3, t) \), \( t \geq 1 \), then every representation \( \{\sigma_i, \tau_i : i = 0, 1, 2\} \) of \( O \) satisfies the following:

(i) For \( i = 0, 1, 2 \), \( \sigma_i \) and \( \tau_i \) are parallel.

(ii) For all \( x \in S^{-1} \setminus \{a_0(x), a_1(x), a_2(x), \tau_0(x), \tau_1(x)\} \) \( = 2 + S \).

Here (i) follows from the simplicity of the array, and (ii) is a consequence of the array’s having index 2.

Finally, we need one more observation:

**Lemma 5.4** Let \(|S| = 3\). If \( a_1, a_2, a_3, b_1, b_2, b_3 \in S \) such that \( a_i \neq b_i \forall i \) and \( \{a_1, a_2, a_3, b_1, b_2, b_3\} \) \( = 2 + S \), then \( i \neq j \Rightarrow S \subset \{a_i, a_j, b_i, b_j\} \).

We are now ready to prove the main theorems.

**Theorem 5.5** Every simple orthogonal array of the form \( OA(2(3^t), t+1, 3, t) \), \( t \geq 1 \), has a representation \( \{\sigma_0, \sigma_1, \sigma_2, \tau_0, \tau_1, \tau_2\} \subseteq F_{t-1} \).

**Proof of Theorem 5.5.** Let \( O \) be a simple orthogonal array of the form \( OA(2(3^t), t+1, 3, t) \). We will show that we can construct a partition of \( O \) such that the corresponding representation of \( O \) is in \( F_{t-1} \).

Let \( O = A_0 + A_1 + A_2 + B_0 + B_1 + B_2 \) be a partition of \( O \) as in (3), and let \( \{\sigma_0, \sigma_1, \sigma_2, \tau_0, \tau_1, \tau_2\} \) be the corresponding representation. If \( \sigma_0, \sigma_1, \sigma_2, \tau_0, \tau_1, \tau_2 \subseteq F_{t-1} \), we are done. Otherwise, we will show that by interchanging columns between \( A \) and \( B \) it is possible to construct a new partition \( O = A_0 + A_1 + A_2 + B_0 + B_1 + B_2 \) such that \( \sigma_i, \tau_i \in F_{t-1} \forall i \in \{0, 1, 2\} \).

The exchange of columns between \( A_i \) and \( B_i \) is governed by the following rules:

1. We will only exchange columns between \( A \) and \( B \), and the only exchanges permitted are between columns that share all but the last component.

2. For each \( i \), we will execute the exchange of columns in \( t-1 \) stages, one stage for each argument of \( \sigma_i \) and \( \tau_i \). We will go through the \( t-1 \) arguments “backwards”: at the \( k^{th} \) stage we examine the \( (t-k)^{th} \) sections of \( \sigma_i \) and \( \tau_i \), where \( k \) runs from 1 through \( t-1 \). Since there are \( 3^{t-2} \) ways to fix all but one argument, there are that many steps at each stage.

3. In each step in the \( k^{th} \) stage, we fix all but the \( (t-k)^{th} \) argument of \( \sigma_i \) and \( \tau_i \). This determines columns of \( O \) having prescribed values in all but the \( (t-k-1)^{st} \) and the last row. Among these columns, at most one exchange will occur. Whether or not an exchange is necessary is determined using Remark 3.11, as described below.
Every exchange of columns will cause a redefinition of $\sigma_i$ and $\tau_i$. This means that upon completion of all stages, we will have constructed “new” sets $A_i$ and $B_i$ with corresponding “new” functions $\sigma_i$, $\tau_i$, which – as we will show – are in $F_{t-1}$. Recall that $\sigma_i, \tau_i \in F_{t-1}$ if and only if all sections of $\sigma_i$ and $\tau_i$ are in $F_1$ (i.e. are one-to-one).

Fix $i \in \{0, 1, 2\}$. For this $i$, we will show how the columns within a step are determined. Then we will use Remark 3.11 to decide whether an exchange of columns takes place, which columns are to be switched if an interchange occurs, and how this redefines $\sigma_i$ and $\tau_i$.

Suppose we are at the $k$th stage. We fix $y \in S^{t-k-1}$ and $z \in S^{k+1}$ and consider the sections $\sigma_i(y, z)$ and $\tau_i(y, z)$. Since $O$ has index 2, this determines six columns, namely two columns of the form $(i, y, j, z, *)^T$ for each $j \in S$, as every $t$-tuple occurs exactly twice in $O$. By construction, these are the columns

\[
\begin{pmatrix}
i \\
y \\
\sigma_i(y, j, z) \\
\text{and} \\
i \\
y \\
\tau_i(y, j, z)
\end{pmatrix}
\]

where $j \in S$.

Let $\alpha_j = \sigma_i(y, j, z)$, $\beta_j = \tau_i(y, j, z)$; these are the last components of the six columns in (4). By the assumption of simplicity, $\alpha_j \neq \beta_j \forall j$. Again using the fact that the array has index 2, notice that every $t$-tuple from rows $1, \ldots, t-k-1, t-k+1, \ldots, t+1$ occurs twice in $O$, so if $i, y, z$ are fixed, we must have $\{\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\} = 2^S$.

Now let $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, $\beta = (\beta_0, \beta_1, \beta_2)$ and pick $\pi$ and $\gamma$ as in Lemma 3.10. If $\pi = \alpha$, no exchange of columns takes place. Otherwise, there is exactly one $\beta_j \in \{0, 1, 2\}$ such that $\pi_j = \beta_j$. In that case, we interchange the column $(i, y, j, z, \alpha_j)^T$ from $A_i$ with the column $(i, y, j, z, \beta_j)^T$ from $B_i$. This means we redefine the sets $A_i$ and $B_i$, and consequently we redefine $\sigma_i(y, j, z)$ as $\tau_i(y, j, z)$ and vice versa; when $x \neq (y, j, z)^T$, $\sigma_i(y, j, z)$ and $\tau_i(y, j, z)$ remain unchanged. This means we get $\sigma_i(y, \cdot, z) = \pi(\cdot)$ and $\tau_i(y, \cdot, z) = \gamma(\cdot)$. This concludes this step.

By construction, the newly defined sections $\sigma_i(y, \cdot, z)$ and $\tau_i(y, \cdot, z)$ are one-to-one. Since $y$ and $z$ are arbitrary, and since different steps involve disjoint sets of columns, all $k$th sections of $\sigma_i$ and $\tau_i$ are one-to-one once the $k$th stage is completed.

The question is: In carrying out the $k$th stage, are we undoing any results from previous stages? The answer is no, as we will prove now.

Claim: After all exchanges in the $k$th stage are completed, the functions $\sigma_i$ and $\tau_i$ obtained satisfy the property that all their $(t-k-n)^{th}$ sections ($n = 0, \ldots, k+1$) are one-to-one.

Proof of Claim. We will use finite induction to prove the claim.

$k = 1$: Suppose stage 1 is completed. Within this stage, two different steps involve disjoint sets of columns. Thus at the end of the stage, all $(t-1)^{st}$ sections of $\sigma_i$ and $\tau_i$ are one-to-one.

Induction Step: Assume the claim is true for some $k - 1$ ($\leq t - 2$). We have to show that after completion of the $k$th stage, all $(t-m)^{th}$ sections of $\sigma_i$ and $\tau_i$ are one-to-one for all $1 \leq m \leq k$.

We first prove the claim when $m = k$, since this is easiest case. Within the $k$th stage, the sets of columns arising in two different steps are disjoint. Thus all $k$th sections of $\sigma_i$ and $\tau_i$ are one-to-one once the $k$th stage is completed.

We will now prove the claim for $1 \leq m < k$.

During the first $k - 1$ stages, we have carefully constructed functions $\sigma_i$ and $\tau_i$ whose $(t-m)^{th}$ sections are all one-to-one for all $m < k$. The concern is that in carrying out stage $k$, we may be undoing those results. We need to answer this question: How are the $(t-m)^{th}$ sections of $\sigma_i$ and $\tau_i$ affected during stage $k$?
Remark 3.11 that transforms both to be permutations. Thus we need to carry out an exchange of columns according to the injectivity of the $(\beta)$ could have any of the first six patterns as described in Remark 3.11. In order to achieve \{$\sigma, \tau_\ell$\} and \{$\sigma, \tau_\ell$\} are in the same parallel class. Using Lemma 3.8, this implies \[\sigma(\cdot, \ell) = \sigma_i(y, j, v, \ell, w).\]

To investigate completely how the $(t-m)^{th}$ sections of $\sigma_i$ and $\tau_i$ determined by $y, j, v, w$ are affected when an exchange of columns takes place, we have to examine what happens in the steps when $\ell = 0$, $\ell = 1$, and $\ell = 2$. Thus we have to look at three steps. We will show the following:

Claim: If an exchange of columns takes place when $j = a$, then this forces the exchange of columns with $j = a$ when $\ell = 1$ and $\ell = 2$ also.

Proof of claim. For ease of reading, we will use the notation

\[
\sigma(j, \ell) = \sigma_i(y, j, v, \ell, w), \quad \tau(j, \ell) = \tau_i(y, j, v, \ell, w)
\]

for $j, \ell \in S$, keeping $y, v, w$ fixed.

First, we need some preliminary observations. Let $a \in S$. Since the array is simple, $\sigma(a, \cdot)$ and $\tau(a, \cdot)$ are parallel. Let $b \in S$ such that $b \neq a$. Using the fact that the array has index 2 and applying Lemma 5.4, we have

\[
\forall x \in S, \quad S \subseteq \{\sigma(a, x), \tau(a, x), \sigma(b, x), \tau(b, x)\}. \tag{6}
\]

Moreover, $\sigma(a, \cdot), \tau(a, \cdot), \sigma(b, \cdot), \tau(b, \cdot)$ are $(t-m)^{th}$ sections of $\sigma_i$ and $\tau_i$, respectively, where $m < k$. Thus by the induction hypothesis, $\sigma(a, \cdot), \tau(a, \cdot), \sigma(b, \cdot), \tau(b, \cdot)$ are one-to-one (that is, they are in $F_1$). We can therefore apply Proposition 3.9 with $n = 1, f(\cdot) = \sigma(a, \cdot), g(\cdot) = \tau(a, \cdot), \sigma(\cdot) = \sigma(b, \cdot), \tau(\cdot) = \tau(b, \cdot)$ to conclude that $\sigma(a, \cdot), \tau(a, \cdot), \sigma(b, \cdot), \tau(b, \cdot)$ are in the same parallel class. Using Lemma 3.8, this implies $|\{\sigma(a, \cdot), \tau(a, \cdot)\} \cap \{\sigma(b, \cdot), \tau(b, \cdot)\}| = 1$, and

\[
\sigma(b, \cdot) \notin \{\sigma(a, \cdot), \tau(a, \cdot)\} \quad \Rightarrow \quad \sigma(b, \cdot) \text{ is parallel to both } \sigma(a, \cdot) \text{ and } \tau(a, \cdot). \tag{7}
\]

We will now show that one exchange of columns forces two others. Recall that $y, v, w$ are fixed.

Without loss of generality, suppose an exchange takes place in the step when $\ell = 0$. Thus consider the 6 columns whose last entries are given by the components of $\alpha = (\sigma(0, 0), \sigma(1, 0), \sigma(2, 0))$ and $\beta = (\tau(0, 0), \tau(1, 0), \tau(2, 0))$. Since the array is simple, $\sigma(\cdot, 0)$ and $\tau(\cdot, 0)$ are parallel. Therefore $\alpha_j \neq \beta_j \forall j$. Moreover, since the array has index 2, \{\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\} $\leq 2$ $\ast_S$. Thus the conditions of Lemma 3.10 are satisfied, and $\alpha$ and $\beta$ could have any of the first six patterns as described in Remark 3.11. In order to achieve the injectivity of the $(t-k)^{th}$ sections of $\sigma_i$ and $\tau_i$ determined by $\ell = 0$, we need $\alpha$ and $\beta$ both to be permutations. Thus we need to carry out an exchange of columns according to Remark 3.11 that transforms $\alpha$ and $\beta$ into two parallel permutations $\pi$ and $\gamma$. 

\[
\begin{bmatrix}
  i \\
  y \\
  j \\
  v \\
  \ell \\
  w \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  i \\
  y \\
  j \\
  v \\
  \ell \\
  w \\
\end{bmatrix}
\]
We will only carry out one case in detail, namely the case where \( \alpha = (a, a, b), \beta = (c, b, c) \). That is, we assume
\[
\sigma(0, 0) = \sigma(1, 0) \quad \text{and} \quad \tau(0, 0) = \tau(2, 0).
\] (8)
Consider now the step when \( \ell = 1 \). This specifies six columns whose last entries are given by the components of \( \alpha' = (\sigma(0, 1), \sigma(1, 1), \sigma(2, 1)) \) and the components of \( \beta' = (\tau(0, 1), \tau(1, 1), \tau(2, 1)) \). Using (7) and (8), we must have that \( \sigma(0, \cdot) = \sigma(1, \cdot) \) and \( \tau(0, \cdot) \) and \( \tau(1, \cdot) \) are parallel. In particular, \( \sigma(0, 1) = \sigma(1, 1) \). Since the array has strength 2, and invoking condition (6), we also have \( \tau(0, 1) = \tau(2, 1) \). Once again, \( \alpha' \) is of the form \( \alpha' = (a', a', b') \) and \( \beta' \) is of the form \( \beta' = (c', b', c') \). Finally, the argument for the step when \( \ell = 2 \) is identical, so here again, \( \alpha'' = (a'', a'', b'') \) and \( \beta'' \) is of the form \( \beta'' = (c'', b'', c'') \), where \( \alpha'' = (\sigma(0, 2), \sigma(1, 2), \sigma(2, 2)) \) and \( \beta'' = (\tau(0, 2), \tau(1, 2), \tau(2, 2)) \).

In all three steps, the triples labeled \( \alpha \) and \( \beta \) have the same form. Thus we have to make the same exchange in all three steps. According to Remark 3.11, in each step we exchange the columns corresponding to the first component of \( \alpha \) and \( \beta \). Therefore an exchange takes place between the columns \((i, y, 0, v, \ell, w, \sigma(0, \ell))'\) and \((i, y, 0, v, \ell, w, \tau(0, \ell))'\) for each \( \ell \in S \). In our original notation, we exchange the column \((i, y, 0, v, \ell, w, \sigma_i(y, 0, v, \ell, w))'\) with the column \((i, y, 0, v, \ell, w, \tau_i(y, 0, v, \ell, w))'\) for all \( \ell \in S \). Thus in case 1, when an exchange when \( \ell = 0 \) is necessary for \( j = 0 \), we are forced to exchange columns having \( j = 0 \) when \( \ell = 1 \) and \( \ell = 2 \) also. This proves the claim for case 1.

That one exchange of columns in a stage forces two others means the following: An exchange occurring when \( \ell = 0 \) implies that we have to redefine \( \sigma_i(x) \) to be \( \tau_i(x) \) and vice versa at the three points \( x = (y, 0, v, \ell, x) \), \( \ell = 0, 1, 2, \). Now in these three steps we redefine not only the sections \( \sigma_i(y, 0, v, \ell, w) \) and \( \tau_i(y, 0, v, \ell, w) \) for \( \ell = 0, 1, 2 \), but also the sections \( \sigma_i(y, 0, v, \cdot, w) \) and \( \tau_i(y, 0, v, \cdot, w) \). In particular, we redefine the function \( \sigma_i(y, 0, v, \cdot, w) \) as \( \tau_i(y, 0, v, \cdot, w) \) and vice versa. These functions are \( m^\text{th} \) sections of \( \sigma_i \) and \( \tau_i \) and by the induction hypothesis were thus one-to-one at the beginning of this \( (k^\text{th}) \) stage. Therefore the redefined sections \( \sigma_i(y, 0, v, \cdot, w) \) and \( \tau_i(y, 0, v, \cdot, w) \) are again one-to-one.

Note that all other cases corresponding to the different forms \( \alpha \) and \( \beta \) can take are equivalent to the case demonstrated above.

Since \( y, v, w \) were chosen arbitrarily, the “new” functions \( \sigma_i \) and \( \tau_i \) constructed in this stage have the property that all \((t-m)^\text{th}\) sections of \( \sigma_i, \tau_i \) are one-to-one. Thus all \((t-m)^\text{th}\) sections of the functions \( \sigma_i \) and \( \tau_i \) constructed in stage \( k \) are one-to-one for all \( 1 \leq m < k \). This concludes the induction and proves the claim.

We have thus shown that we can interchange columns between \( A_i \) and \( B_i \) so that all \( k^\text{th} \) sections of the resulting functions \( \sigma_i \) and \( \tau_i \) are in \( F_1 \) for all \( 1 \leq k \leq t - 1 \). Therefore, we can rearrange columns so that \( \sigma_i, \tau_i \in \sigma_1 \). Since \( i \) was arbitrary, this is true for all \( i = 0, 1, 2, \) that is, the functions \( \sigma_0, \sigma_1, \sigma_2, \tau_0, \tau_1, \tau_2 \) obtained by the exchange of columns described above are all in \( F_{t-1} \).

Thus \( O \) can be represented by a set \( \{\sigma_0, \sigma_1, \sigma_2, \tau_0, \tau_1, \tau_2\} \subseteq F_{t-1} \).

**Theorem 5.6** Let \( S \) be the set of all simple arrays of the form \( OA(2(3^t), t + 1, 3, t) \), and let \( R \) be the set of all representations of such arrays in \( F_{t-1} \). Then \( |S| = |R| \).

**Proof.** Let \( D \) be the set of all simple decomposable arrays of the form \( OA(2(3^t), t + 1, 3, t) \) and \( R \) the set of all representations of simple arrays of type \( OA(2(3^t), t + 1, 3, t) \) in \( F_{t-1} \). Then \( D \subseteq S \), and by Remark 5.2, there exists a map from \( R \) onto \( S \). Therefore we have \( |D| \leq |S| \leq |R| \).

We will show that \( |R| = |D| = 3(2^t) \).

To find \( |R| \), we apply Lemma 5.3 with \( \lambda = 2 \) and count the number of ways in which we can construct a set \( \{\sigma_0, \sigma_1, \sigma_2, \tau_0, \tau_1, \tau_2\} \subseteq F_{t-1} \) such that

(i) \( \sigma_i \) and \( \tau_i \) are parallel for \( i = 0, 1, 2, \) and
(ii) \( \forall x \in S^{t-1}, \{ \sigma_0(x), \sigma_1(x), \sigma_2(x), \tau_0(x), \tau_1(x), \tau_2(x) \} = 2 * S \).

Now, by Lemma 3.5 and Lemma 3.8, there are \( 3(2^t-1) \) ways to select parallel functions \( \sigma_0, \tau_0 \in F_{t-1} \). Given such \( \sigma_0 \) and \( \tau_0 \), Lemma 3.8 implies that there are exactly two ways to choose \( \{ \sigma_1, \sigma_2, \tau_1, \tau_2 \} \) so that \( \sigma_i \) and \( \tau_i \) are parallel for all \( i \). Moreover, we have \( \{ \sigma_0(x), \sigma_1(x), \sigma_2(x), \tau_0(x), \tau_1(x), \tau_2(x) \} = 2 * S \forall x \). Thus, by Lemma 5.4, \( S \subset \{ \sigma_1(x), \sigma_2(x), \tau_1(x), \tau_2(x) \} \) for all \( x \). By Lemma 5.3, the set \( \{ \sigma_0, \sigma_1, \sigma_2, \tau_0, \tau_1, \tau_2 \} \) is a representation of a simple orthogonal array of form \( OA(2(3^t), t + 1, 3, t) \). Thus \( |R| = 3(2^t) \).

Next, let \( O \) be a simple decomposable array of form \( OA(2(3^t), t + 1, 3, t) \). Then \( O = O_1 + O_2 \), where the \( O_i \) are orthogonal arrays of form \( OA(3^t, t + 1, 3, t) \). By Corollary 3.4, \( O_i \) and \( O_2 \) are regular, that is, \( O_i = L_i + v_i \), where \( L_i \) is a subspace of the vector space \( \mathbb{Z}_3 \). Therefore \( O_i \) is the solution set of a system of linear equations in \( t + 1 \) variables over \( \mathbb{Z}_3 \). Since \( O_i \) has cardinality \( 2^t \) and strength \( t \), it must be the solution set to a single equation whose coefficients are all nonzero.

Suppose \( O_i = \{ x \in \mathbb{Z}_3^{t+1} : c'x = a \} \) and \( O_2 = \{ x \in \mathbb{Z}_3^{t+1} : d'x = b \} \), where \( c = (c_1, c_2, \ldots, c_t) \), \( d = (d_1, d_2, \ldots, d_t) \), \( c_i, d_i \neq 0 \forall i \), and \( a, b \in \mathbb{Z}_3 \). Since \( O \) is simple, we have \( O_1 \cap O_2 = \emptyset \). This is equivalent to the conditions \( c = d \) and \( a \neq b \). There are \( 2^t \) choices for \( c \) and \( 3 \) choices for \( \{ a, b \} \) (as the order of the sets \( O_1 \) and \( O_2 \) in the array doesn’t matter). Thus \( |D| = 3(2^t) \). ■

**Corollary 5.7** Every simple orthogonal array of the form \( OA(2(3^t), t + 1, 3, t) \) has a unique representation in \( F_{t-1} \) and is (completely and regularly) decomposable.

**Proof.** By Theorem 5.6 there are as many simple arrays of the form \( OA(2(3^t), t + 1, 3, t) \) as there are representations of simple arrays of the same type in \( F_{t-1} \). Further, as noted in Remark 5.2, every array of the form \( OA(2(3^t), t + 1, 3, t) \) has at least one representation, and each such array has a representation in \( F_{t-1} \) by Theorem 5.5. Therefore, every simple orthogonal array of the form \( OA(2(3^t), t + 1, 3, t) \) has a unique representation \( \{ \sigma_0, \sigma_1, \sigma_2, \tau_0, \tau_1, \tau_2 \} \subset F_{t-1} \).

Next we will show the following: If we define \( A_i \) and \( B_i \) as in (3), where \( \{ \sigma_0, \sigma_1, \sigma_2, \tau_0, \tau_1, \tau_2 \} \) denotes the unique representation of \( O \) in \( F_{t-1} \), and if we let \( A = A_0 + A_1 + A_2 \) and \( B = B_0 + B_1 + B_2 \), then \( O = A + B \) is a complete decomposition of \( O \).

To do so, we define the function \( f_A : \mathbb{Z}_3^{t} \rightarrow \mathbb{Z}_3 \) as follows: If \( x = (i_1, x_2, \ldots, x_t) \in \mathbb{Z}_3^{t} \), then \( f_A(x) = \sigma_i(x_2, \ldots, x_t) \). It is easily verified that \( f_A \in F_t \), and thus \( A \) is an orthogonal array of the form \( OA(3^t, t + 1, 3, t) \). Similarly, using the function \( f_B(x) = \tau_i(x_2, \ldots, x_t) \) whenever \( x = (i_1, x_2, \ldots, x_t) \in \mathbb{Z}_3^{t} \), we find that \( B \) is an array of the form \( OA(3^t, t + 1, 3, t) \) as well. Therefore \( A \) and \( B \) are both of index 1 , and \( O = A + B \) is a complete decomposition of \( O \).

It is easily verified that \( A \) and \( B \) as defined above by the functions \( f_A \) and \( f_B \) are the only subsets of \( O \) that are arrays of strength \( t \) and index 1 . Thus the decomposition \( O = A + B \) is unique. ■

## 6 Outlook

We have shown that while not every orthogonal array is completely decomposable, or even decomposable, every simple orthogonal array of form \( OA(N, t + 1, 3, t) \) is completely decomposable.

When the index of a completely decomposable orthogonal array of form \( OA(N, t + 1, 3, t) \) is greater than 3, the array is not simple and may or may not have a unique decomposition (for example, if such an array contains an orthogonal array of index 3, the decomposition is not unique).

Orthogonal arrays of the form \( OA(N, t + 1, 3, t) \) that are not completely decomposable have not yet been investigated. It is not yet known whether such arrays have any kind of
identifiable algebraic structure, similar to the regularity of index 1 arrays, that would aid us in identifying the different types.

When \( s > 3 \), the situation is even more complex. Unlike in the case when \( s \leq 3 \), there now exist nonregular arrays of index 1 (see [4]). Thus we would expect the problem of identifying (completely) decomposable orthogonal arrays to be even more complicated. One might start by developing a theory that allows us to determine whether a given orthogonal array is completely regular decomposable.

7 Acknowledgements

I wish to thank Jay H. Beder, whose suggestions proved invaluable during the preparation of this paper. I am also grateful to the referee for some very helpful comments.

References


