EXPONENTIAL STABILITY IN NONLINEAR DIFFERENCE EQUATIONS

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Abstract. Non-negative definite Lyapunov functionals are employed to obtain sufficient conditions that guarantee exponential stability of the zero solution of a nonlinear discrete system. The theory is illustrated with several examples.

1. Introduction

Recently, many researchers have investigated the asymptotic stability of the zero solution of Volterra difference equations. Islam and Raffoul [5], [6], and Khandaker and Raffoul [8] investigated the asymptotic stability of the zero solution of a linear Volterra discrete system by expressing the solution in terms of the resolvent matrix $R(n, s)$. However, a major limitation of this procedure is that the resolvent matrix is an abstract term, which requires a summability condition. Eloe, Islam and Raffoul [4] employed the notion of total stability and obtained conditions for the uniform asymptotic stability of the zero solution of a perturbed Volterra discrete system. Again, they had to require that the resolvent be summable. In [3], Elaydi and Murakami studied the exponential stability of discrete Volterra systems imposing a summability condition on $R(n, s)$. Showing that the resolvent matrix $R(n, s)$ is summable is not an easy task and for more on this we refer the reader to [4] and [8]. For some recent results on the stability of the zero solution, we refer the interested reader to [1] [2], [6], [7], [9], [11], [12], [13], [14], [15] and the references therein.

To avoid using the resolvent matrix approach, we make use of non-negative definite Lyapunov functionals by extending the work of [10] to study the exponential stability of the zero solution of the nonlinear discrete system

\begin{equation}
\begin{aligned}
x(n+1) &= f(n, x(n)), \quad n \geq 0, \\
x(n_0) &= x_0, \quad n_0 \geq 0
\end{aligned}
\end{equation}

where $x(n) \in \mathbb{R}^K$, $f(n, x(n)) : \mathbb{Z}^+ \times \mathbb{R}^K \to \mathbb{R}^K$ is a given nonlinear function satisfying $f(n, 0) = 0$ for all $n \in \mathbb{Z}^+$. We assume that $f(n, x)$ has the required conditions so that solutions exist for all $n \geq 0$. For more on the

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existence of solutions of (1.1) we refer the reader to [15]. Throughout this paper, \( \mathbb{R}^K \) is the \( K \)-dimensional Euclidean vector space; \( \mathbb{Z}^+, \mathbb{R}^+ \) are the sets of all non-negative integers, non-negative numbers, respectively; \( ||x|| \) is the Euclidean norm of the vector \( x(n) \in \mathbb{R}^K \). From this point forward, when a function is written without its argument, the argument is assumed to be \( n \).

2. Exponential Stability

In this section we use Lyapunov type functions and establish sufficient conditions to obtain exponential stability and uniform exponential stability of the zero solution of (1.1).

**Definition 2.1** The zero solution of system (1.1) is said to be exponentially stable if any solution \( x(n, n_0, x_0) \) of (1.1) satisfies

\[
||x(n, n_0, x_0)|| \leq C(||x_0||, n_0) a^{-\delta(n-n_0)}, \quad \text{for all } n \geq n_0,
\]

where \( a \) is constant with \( a > 1 \), \( C : \mathbb{R}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{R}^+ \), and \( \delta \) is a positive constant. The zero solution of (1.1) is said to be uniformly exponentially stable if \( C \) is independent of \( n_0 \).

**Theorem 2.2** Let \( a \) be a constant with \( a > 1 \). Let \( D \subset \mathbb{R}^K \) be an open set containing the origin, and let \( V(n, x) : \mathbb{Z}^+ \times D \rightarrow \mathbb{R}^+ \) be a given function satisfying

\[
\lambda_1 ||x||^p \leq V(n, x) \leq \lambda_2 ||x||^q, \tag{2.1}
\]

and

\[
\Delta V(n, x) \leq -\lambda_3 ||x||^r + ka^{-\delta n}, \tag{2.2}
\]

for some positive constants \( \lambda_1, \lambda_2, \lambda_3, p, q, r, k \) and \( \delta \). Moreover, if for some positive constants \( \alpha \) and \( \gamma \),

\[
0 < \frac{\lambda_3}{\lambda_2^{r/q}} \leq \alpha < 1 \tag{2.3}
\]

such that

\[
V(n, x) - V^{r/q}(n, x) \leq \gamma a^{-\delta n} \tag{2.4}
\]

with

\[
\delta > -\frac{\ln(1 - \lambda_3/\lambda_2^{r/q})}{\ln(a)}, \tag{2.5}
\]

then the zero solution of (1.1) is uniformly exponentially stable.

**Proof** First note that in view of (2.3), the constant \( \delta \), which is given by (2.5) is positive. Taking the difference of the function \( V(n, x)a^{M(n-n_0)} \) with

\[
M = -\frac{\ln(1 - \lambda_3/\lambda_2^{r/q})}{\ln(a)}
\]

we have

\[
\Delta \left( V(n, x)a^{M(n-n_0)} \right) = \left[ V(n + 1, x)a^{M} - V(n, x) \right]a^{M(n-n_0)}.
\]
For \( x \in D \), using (2.2) we get
\[
\Delta \left( V(n, x) a^{M(n-n_0)} \right) \leq \left[ -\lambda_3 ||x||^r a^M + V(n, x) a^M \right] + ka^{-\delta n} a^M - V(n, x) \right] a^{M(n-n_0)}. \tag{2.6}
\]

From condition (2.1) we have \( ||x||^q \geq V(n, x)/\lambda_2 \). Consequently, \(-||x||^r \leq -[V(n,x)]^{r/q} \). Thus, inequality (2.6) becomes
\[
\Delta \left( V(n, x) a^{M(n-n_0)} \right) \leq \left[ -a^M (\lambda_3/\lambda_2^{r/q}) V^{r/q}(n, x) + V(n, x) a^M \right. \\
+ ka^{-\delta n} a^M - V(n, x) \right] a^{M(n-n_0)} \\
= \left[ -a^M (\lambda_3/\lambda_2^{r/q}) V^{r/q}(n, x) + (a^M - 1)V(n, x) \right] + ka^{-\delta n} a^M \right] a^{M(n-n_0)}.
\]

Since \( M = -\frac{\ln(1-\lambda_3/\lambda_2^{r/q})}{\ln(a)} \), we have \( a^M - 1 = a^M (\lambda_3/\lambda_2^{r/q}) \). Thus, the above inequality reduces to
\[
\Delta \left( V(n, x) a^{M(n-n_0)} \right) \leq \left[ (a^M - 1) \left( V(n, x) - V^{r/q}(n, x) \right) \right. \\
+ ka^{-\delta n} a^M \right] a^{M(n-n_0)} \tag{2.7}
\]

By invoking condition (2.4), inequality (2.7) takes the form
\[
\Delta \left( V(n, x) a^{M(n-n_0)} \right) \leq \left( (a^M - 1) \gamma + ka^M \right) a^{-\delta n} a^{M(n-n_0)} \\
\leq \left( (a^M - 1) \gamma + ka^M \right) a^{-\delta n + \delta n_0} a^{M(n-n_0)} \\
= L a^{(M-\delta)(n-n_0)},
\]

where \( L = (a^M - 1) \gamma + ka^M \). Summing the above inequality from \( n_0 \) to \( n - 1 \) we obtain,
\[
V(n, x) a^{M(n-n_0)} - V(n_0, x_0) \leq L a^{-(M-\delta)n_0} \sum_{s=n_0}^{n-1} a^{(M-\delta)s} \\
= \frac{L a^{-(M-\delta)n_0}}{a^{(M-\delta)} - 1} \left[ a^{(M-\delta)n} - a^{(M-\delta)n_0} \right] \\
= \frac{L}{a^{(M-\delta)} - 1} \left[ a^{(M-\delta)(n-n_0)} - 1 \right].
\]

Since \( M < \delta \) and \( V(n_0, x_0) \leq \lambda_2 ||x_0||^q \), the above inequality reduces to
\[
V(n, x) a^{M(n-n_0)} \leq \lambda_2 ||x_0||^q + \frac{L}{1-a^{(M-\delta)}}.
\]

Set \( B(||x_0||) = \lambda_2 ||x_0||^q + \frac{L}{1-a^{(M-\delta)}} \). Then
\[
V(n, x) \leq B(||x_0||) a^{-M(n-n_0)}. \tag{2.8}
\]
From condition (2.1), we have
\[ \|x\| \leq \left\{ \frac{V(n, x)}{\lambda_1} \right\}^{1/p}. \tag{2.9} \]
Combining (2.8) and (2.9), we arrive at
\[ \|x\| \leq \left\{ \frac{B(\|x_0\|)}{\lambda_1} \right\}^{1/p} a^{-\frac{M}{p}(n-n_0)} = C(\|x_0\|) a^{-\frac{M}{p}(n-n_0)}. \]
Hence, the zero solution of (1.1) is uniformly exponentially stable.

**Example 2.3** Consider the nonlinear difference equation
\[ x(n+1) = \sigma x(n) + Rx^{1/3}(n)a^{-ln}, \tag{2.10} \]
a > 1 and \( l \) are constants with \( l > -\frac{1}{3} \ln\left(1 - \lambda_3^{\lambda_3/\lambda_2}\right) \), where \( \lambda_1 = \lambda_2 = 1, \lambda_3 = 1 - \left(\sigma^2 + \frac{4}{3} |\sigma||R| + \frac{R^2}{3}\right) \), \( p = 2 \), and \( q = r = 2 \). If
\[ \sigma^2 + \frac{4}{3} |\sigma||R| + \frac{R^2}{3} < 1, \]
then the zero solution of (2.10) is uniformly exponentially stable.

To see this, let \( V(n, x) = x^2(n) \). By calculating \( \triangle V(n, x) \) along the solutions of (2.10), we obtain
\[ \triangle V(n, x) = x^2(n+1) - x^2(n) \]
\[ = (\sigma x + Rx^{1/3}a^{-ln})^2 - x^2 \]
\[ \leq \sigma^2 x^2 + 2|\sigma||R|x^{4/3}a^{-ln} + R^2 x^{2/3}a^{-2ln} - x^2 \]

To further simplify \( \triangle V(n, x) \), we make use of Young’s inequality \( wz < \frac{w^e}{e} + \frac{z^f}{f} \) with \( 1/e + 1/f = 1 \). Thus, for \( e = 3/2 \) and \( f = 3 \), we have
\[ 2|\sigma||R|x^{4/3}a^{-ln} \leq 2|\sigma||R|\left[ (x^{4/3})^{3/2} + \frac{a^{-3ln}}{3} \right] \]
\[ = 4|\sigma||R|x^{2/3} + \frac{2}{3} |\sigma||R|a^{-3ln}. \]
Similarly, if we let \( e = 3 \) and \( f = 3/2 \), we have
\[ R^2 x^{2/3}a^{-2ln} \leq R^2 \left[ (x^{2/3})^{3/2} + \frac{a^{-3ln}}{3} \right] \]
\[ = \frac{R^2 x^2}{3} + \frac{2}{3} R^2 a^{-3ln}. \]
Thus,
\[ \triangle V(n, x) \leq \left[ \sigma^2 + \frac{4}{3} |\sigma||R| + \frac{R^2}{3} - 1 \right] x^2 + \left[ \frac{2}{3} |\sigma||R| + \frac{2}{3} R^2 \right] a^{-3ln} \]
\[ \leq - \left[ 1 - \left( \sigma^2 + \frac{4}{3} |\sigma||R| + \frac{R^2}{3} \right) \right] x^2 + \left[ \frac{2}{3} |\sigma||R| + \frac{2}{3} R^2 \right] a^{-3ln}. \]
One can easily check that conditions (2.1)-(2.5) of Theorem 2.2 are satisfied with 
\[3l = \delta \text{ and } k = \frac{2}{3}|\sigma||R| + \frac{2}{3}R^2.\] Hence the zero solution of (2.10) is uniformly exponentially stable.

In the next theorem we show that the zero solution of (1.1) is exponentially stable.

**Theorem 2.4** Let \(a\) be a constant with \(a > 1\). Let \(D \subset \mathbb{R}^k\) be an open set containing the origin, and let \(V(n, x) : \mathbb{Z}^+ \times D \to \mathbb{R}^+\) be a given function satisfying

\[\lambda_1(n)||x||^p \leq V(n, x) \leq \lambda_2(n)||x||^q,\]  

(2.11)

and

\[\triangle V(n, x) \leq -\lambda_3(n)||x||^r + ka^{-\delta n},\]  

(2.12)

for some positive constants \(p, q, r, k, \delta, \) and positive functions \(\lambda_1(n), \lambda_2(n)\) and \(\lambda_3(n),\) where \(\lambda_1(n)\) is a non-decreasing sequence. Moreover, suppose for some positive constants \(\alpha \) and \(\gamma,\)

\[0 < \frac{\lambda_3(n)}{\lambda_2^{r/q}(n)} \leq \alpha < 1\]  

(2.13)

such that

\[V(n, x) - V^{r/q}(n, x) \leq \gamma a^{-\delta n}\]  

(2.14)

with

\[\delta > \inf_{n \in \mathbb{Z}^+} - \frac{\ln(1 - \lambda_3(n)/\lambda_2^{r/q}(n))}{\ln(a)}.\]  

(2.15)

Then the zero solution of (1.1) is exponentially stable.

**Proof** First note that in view of (2.13), \(\delta\) which is given by (2.15) is positive. Taking the difference of the function \(V(n, x)a^{M(n-n_0)}\) with

\[M = \inf_{n \in \mathbb{Z}^+} - \frac{\ln(1 - \lambda_3(n)/\lambda_2^{r/q}(n))}{\ln(a)},\]

we have

\[\Delta \left(V(n, x)a^{M(n-n_0)}\right) = \left[V(n+1, x)a^M - V(n, x)\right]a^{M(n-n_0)}.\]

By a similar argument as in Theorem 2.2 we obtain,

\[V(n, x) \leq B(||x_0||, \lambda_2(n_0))a^{-M(n-n_0)},\]  

(2.16)

where \(B(||x_0||, \lambda_2(n_0)) = \lambda_2(n_0)||x_0||^q + \frac{L}{1-a^{r/q}}.\) From condition (2.11) and the fact that \(\lambda_1(n)\) is non-decreasing we have,

\[||x|| \leq \left\{\frac{V(n, x)}{\lambda_1(n)}\right\}^{1/p} \leq \left\{\frac{V(n, x)}{\lambda_1(n_0)}\right\}^{1/p}.\]  

(2.17)
Combining (2.16) and (2.17) we obtain
\[
||x|| \leq \left\{ \frac{B(||x_0||, \lambda_2(n_0))}{\lambda_1(n_0)} \right\}^{1/p} a^{-\frac{M}{p}(n-n_0)}
= C(||x_0||, n_0) a^{-\frac{M}{p}(n-n_0)}.
\]

Hence, the zero solution of (1.1) is exponentially stable. The next corollary is an immediate consequence of Theorem 2.4.

**Corollary 2.5** Suppose the hypotheses of Theorem 2.4 hold where the non-decreasing condition on \(\lambda_1(n)\) is replaced by
\[
\lambda_1(n) \geq a^{-Nn} \quad \text{for all } n \geq n_0 \geq 0 \text{ with } 0 < N < M,
\]
then the zero solution of (1.1) is exponentially stable.

The next theorem does not require an upper bound on the Lyapunov function.

**Theorem 2.6** Let \(a\) be a constant with \(a > 1\). Let \(D \subset \mathbb{R}^k\) be an open set containing the origin, and let \(V(n, x) : \mathbb{Z}^+ \times D \to \mathbb{R}^+\) be a given function satisfying
\[
\lambda_1 ||x||^p \leq V(n, x), \quad (2.18)
\]
and
\[
\Delta V(n, x) \leq -\lambda_2 V(n, x) + ka^{-\delta n}, 0 < \lambda_2 < 1 \quad (2.19)
\]
for some positive constants \(\lambda_1, \lambda_2, p, k\) and \(\delta\).

Then, the zero solution of (1.1) is exponentially stable.

**Proof** Let \(\epsilon\) be a fixed number satisfying
\[
0 < \epsilon < \min\left\{ \frac{-ln(1-\lambda_2)}{ln(a)}, \delta \right\}.
\]

Note that \(\frac{-ln(1-\lambda_2)}{ln(a)} > 0\), since \(0 < \lambda_2 < 1\) and \(a > 1\). Taking \(\Delta \left( a^{\epsilon n} V(n, x) \right)\) along the solutions of (1.1) and utilizing conditions (2.18) and (2.19), we obtain
\[
\Delta \left( V(n, x)a^{\epsilon n} \right) = a^{\epsilon n} \left[ a^\epsilon (1-\lambda_2) V(n, x) - V(n, x) + ka^{-\delta n} \right] \leq ka^{\epsilon n} a^{-\delta n}.
\]
Thus,

\[ V(n, x) a^{\epsilon n} \leq a^{\epsilon n_0} V(n_0, x_0) + k \sum_{s=n_0}^{n-1} a^{(\epsilon-\delta)s} \]

\[ \leq a^{\epsilon n_0} V(n_0, x_0) + \frac{k}{a^{(\epsilon-\delta)} - 1} [a^{(\epsilon-\delta)n} - a^{(\epsilon-\delta)n_0}] \]

\[ \leq a^{\epsilon n_0} V(n_0, x_0) + \frac{ka^{(\epsilon-\delta)n_0}}{1 - a^{(\epsilon-\delta)}} \]

\[ \leq a^{\epsilon n_0} \left( V(n_0, x_0) + \frac{k}{1 - a^{(\epsilon-\delta)}} \right). \]

Therefore,

\[ |x|^p \leq \frac{1}{\lambda_1} \left( V(n_0, x_0) + \frac{k}{1 - a^{(\epsilon-\delta)}} \right) a^{-\epsilon(n-n_0)}, \text{ for all } n \geq n_0. \]

This completes the proof.

**Example 2.7** Let \( a \) be a constant such that \( a > 1 \) and consider the nonlinear difference equation

\[ x(n+1) = \sigma x + R x^{1/3} + a^{\gamma_1 n} \sin(x), \quad (2.20) \]

where \( \gamma_1 > 0 \) and

\[ a^{-\gamma_2} \left( |\sigma| + \frac{1}{3} \right) < 1. \]

Suppose \( \gamma_2 < M = \frac{-\ln(a^{-\gamma_2}(|\sigma| + \frac{1}{3})]}{\ln(a)} \). If \( \gamma_1 - \gamma_2 \leq -\eta \) for some positive constant \( \eta \), with \( \eta > \frac{-\ln(a^{-\gamma_2}(|\sigma| + \frac{1}{3})]}{\ln(a)} \) and \( \gamma_2 > 0 \), then the zero solution of (2.20) is exponentially stable. To see this, let \( V(n, x) = a^{-\gamma_2 n} |x(n)| \). By calculating \( \Delta V(n, x) \) along the solutions of (2.20), we obtain

\[ \Delta V(n, x) = a^{-\gamma_2 (n+1)} |x(n+1)| - a^{-\gamma_2 n} |x(n)| \]

\[ \leq \left( |\sigma||x| + |R||x|^{1/3} + a^{\gamma_1 n} \right) a^{-\gamma_2 (n+1)} - a^{-\gamma_2 n} |x(n)|. \]

Using Young’s inequality with \( e = 3/2 \) and \( f = 3 \), we obtain

\[ |R||x|^{1/3} \leq \frac{2}{3} |R|^{3/2} + |x|/3. \]
Thus,\[
\begin{align*}
\triangle V(n, x) & \leq a^{-\gamma_2 n} \left[ a^{-\gamma_2 (|\sigma| + \frac{1}{3})} - 1 \right] |x| \\
& + \frac{2}{3} a^{-\gamma_2 (n+1)} |\sigma|^{3/2} a^{-\gamma_2 (n+1) + \gamma_1 n} \\
& \leq a^{-\gamma_2 n} \left[ a^{-\gamma_2 (|\sigma| + \frac{1}{3})} - 1 \right] |x| \\
& + \frac{2}{3} a^{-\gamma_2 (n+1)} a^{\gamma_1 n} |\sigma|^{3/2} a^{-\gamma_2 (n+1) + \gamma_1 n} \\
& = a^{-\gamma_2 n} \left[ a^{-\gamma_2 (|\sigma| + \frac{1}{3})} - 1 \right] |x| \\
& + a^{-\gamma_2 \left( \frac{2}{3} |\sigma|^{3/2} + 1 \right)} a^{(\gamma_1 - \gamma_2) n} \\
& = - \left[ 1 - a^{-\gamma_2 (|\sigma| + \frac{1}{3})} \right] a^{-\gamma_2 n} |x| \\
& + a^{-\gamma_2 \left( \frac{2}{3} |\sigma|^{3/2} + 1 \right)} a^{-\eta n}.
\end{align*}
\]
Thus, the conditions of Corollary 2.5 are satisfied with \( N = \gamma_2, \lambda_1(n) = \lambda_2(n) = a^{-\gamma_2 n}, \lambda_3(n) = 1 - a^{-\gamma_2 (|\sigma| + \frac{1}{3})} a^{-\gamma_2 n}, \ p = q = r = 1, \delta = \eta \) and \( k = a^{-\gamma_2 \left( \frac{2}{3} |\sigma|^{3/2} + 1 \right)} \). Hence the zero solution of (2.20) is exponentially stable.

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