Boundedness And Stability In Nonlinear Discrete Systems With Nonlinear Perturbation

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Abstract

We consider the nonlinear Volterra discrete system with nonlinear perturbation

\[ x(n+1) = A(n) x(n) + \sum_{s=0}^{n} B(n, s) f(s, x(s)) + g(n, x(n)). \]

Our goal is to use Lyapunov functionals to obtain conditions that guarantee all solutions of the above Volterra equation are bounded and derive conditions that ensure asymptotic stability and exponential stability, in the case \( x = 0 \) is a solution.

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1 Introduction

In this paper, we have studied the boundedness of solutions and stability properties of the zero solution of the nonlinear perturbed Volterra discrete system

\[ x(n+1) = A(n) x(n) + \sum_{s=0}^{n} B(n, s) f(s, x(s)) + g(n, x(n)) \quad (1.1) \]

where \( g(n, x(n)) \) and \( f(n, x(n)) \) are \( k \times 1 \) vector functions that are continuous in \( x \) and satisfy

\[ |g(n, x(n))| \leq \lambda_1(n) + \lambda_2(n) |x(n)|, \]

and

\[ |f(n, x(n))| \leq \gamma(n) |x(n)|. \]

We assume \( \gamma(n) \) is positive and bounded, \( 0 \leq \lambda_1(n) \leq M \) and \( 0 < \lambda_2(n) \leq L \) for some positive constants \( M \) and \( L \). Moreover, \( A(n) \) and \( B(n, s) \) are \( k \times k \) matrix functions on
\( \mathbb{Z}^+ \) and \( \mathbb{Z}^+ \times \mathbb{Z}^+ \), where \( \mathbb{Z}^+ \) denotes the set of all nonnegative integers. If \( A = (a_{ij}) \) is a \( k \times k \) real matrix, then we define the norm of \( A \) by

\[
|A| = \max_{1 \leq i \leq k} \left\{ \sum_{j=1}^{k} |a_{ij}| \right\}.
\]

Similarly, for \( x \in \mathbb{R}^k \), \( |x| \) denotes the sup norm of \( x \).

Recently, several authors have studied the behavior of solutions of variant forms of (1.1). Medina [11], [12], [13], Eloe, Islam and Raffoul [5], and Raffoul [14], obtained stability and boundedness results of the solutions of the homogeneous part of (1.1) by means of representing the solution in terms of the resolvent matrix. Eloe and Murakami [5] and Elaydi et al. [4], used the notion of total stability and established results on the asymptotic behavior of the zero solution of (1.1). Their work heavily depended on showing or assuming the summability of the resolvent matrix. However, a major limitation of this procedure is that the resolvent matrix is an abstract term. When \( f(n, x) = x \) and \( |g(n, x(n))| \leq \lambda_2(n) |x(n)| \), it was shown in [5] and [8], that the zero solution of (1.1) is uniformly asymptotically stable provided that \( \sum_{n=0}^{\infty} \lambda_2(n) < \infty \). In this research, we do not assume that \( \sum_{n=0}^{\infty} \lambda_2(n) < \infty \).

We say that \( x(n) = x(n, n_0, \phi) \) is a solution of (1.1) with a bounded initial function \( \phi : [0, n_0] \to \mathbb{R}^k \) if it satisfies (1.1) for \( n > n_0 \) and \( x(j) = \phi(j) \) for \( j \leq n_0 \).

Throughout the paper we write \( x(n) \) for \( x(n, n_0, \phi) \) unless it is otherwise stated.

**Definition 1.1.** Solutions of (1.1) are uniformly bounded if for each \( B_1 > 0 \) there is \( B_2 > 0 \) such that if \( n_0 \geq 0 \), \( \phi : [0, n_0] \to \mathbb{R}^k \) with \( ||\phi(n)|| < B_1 \) on \([0, n_0]\), implies \( |x(n, n_0, \phi)| < B_2 \), for \( n \geq n_0 \) where \( ||\phi|| = \sup |\phi(n)|, 0 \leq n \leq n_0 \).

**Definition 1.2.** Solutions of (1.1) are uniformly ultimately bounded for bound \( B \) if there is a \( B > 0 \) and if for each \( B_1 > 0 \) there exists \( N > 0 \) such that if \( n_0 \geq 0 \), \( \phi : [0, n_0] \to \mathbb{R}^k \) with \( ||\phi(n)|| < B_1 \) on \([0, n_0]\), implies \( |x(n, n_0, \phi)| < B \), for \( n \geq n_0 + N \).

In the case \( x = 0 \) is a solution of (1.1), we state the following definitions.

**Definition 1.3.** The zero solution of (1.1) is stable if for each \( \varepsilon > 0 \), there is a \( \delta = \delta(\varepsilon) > 0 \) such that if \( \phi : [0, n_0] \to \mathbb{R}^k \) with \( ||\phi(n)|| < \delta \) on \([0, n_0]\), implies \( |x(n, n_0, \phi)| < \varepsilon \).

It is uniformly stable (US) if \( \delta \) is independent of \( n_0 \).
Definition 1.4. The zero solution of (1.1) is uniformly asymptotically stable if it is US and there is \( \eta > 0 \) such that for each \( \mu > 0 \) there exists \( N(\mu) > 0 \) independent of \( n_0 \), such that \( |x(n, n_0, \phi)| < \mu \) for all \( n \geq n_0 + N(\mu) \), whenever \( ||\phi(n)|| < \eta \) on \([0, n_0]\).

Definition 1.5. The zero solution of (1.1) is said to be exponentially stable if any solution \( x(n, n_0, x_0) \) of (1.1) satisfies

\[
|x(n, n_0, x_0)| \leq C\left(||\phi||, n_0\right)a^{\delta(n-n_0)}, \quad \text{for all } n \geq n_0,
\]

where \( a \) is constant with \( 0 < a < 1 \), \( C : \mathbb{R}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+ \), and \( \delta \) is a positive constant. The zero solution of (1.1) is said to be uniformly exponentially stable if \( C \) is independent of \( n_0 \).

2 Boundedness and Stability

In this section, we will be using Lyapunov functionals to prove our main results which give an explicit uniform bound on all solutions of (1.1) and then, obtain stability results about the zero solution of (1.1).

Theorem 2.1. Suppose there is a function \( \varphi(n) \geq 0 \) with \( \Delta \varphi(n) \leq 0 \) for \( n \geq 0 \), \( \sum_{n=0}^{\infty} \varphi(n) < \infty \), and \( \Delta_n \varphi(n - s - 1) + |B(n, s)|\gamma(s) \leq 0 \) for \( 0 \leq s < n < \infty \). If for \( n \geq 0 \), \( |A(n)| + |B(n, n)|\gamma(n) + \lambda_2(n) + \varphi(0) \leq 1 - \alpha \) for some \( \alpha \in (0, 1) \), then all solutions of (1.1) are uniformly bounded. Moreover, if \( \lambda_1(n) = 0 \) then the zero solution of (1.1) is uniformly asymptotically stable.

Proof. Define

\[
V(n, x(\cdot)) = |x(n)| + \sum_{s=0}^{n-1} \varphi(n - s - 1)|x(s)|.
\]

(2.1)

Along solutions of (1.1), we have

\[
\Delta V(n, x(\cdot)) = |x(n + 1)| - |x(n)| + \sum_{s=0}^{n} \varphi(n - s)|x(s)| - \sum_{s=0}^{n-1} \varphi(n - s - 1)|x(s)|
\]

\[
\leq |A(n)||x(n)| + \sum_{s=0}^{n} |B(n, s)|\gamma(s)|x(s)| + \lambda_2(n)|x(n)|
\]

\[
+ \lambda_1(n) + \sum_{s=0}^{n} \varphi(n - s)|x(s)| - \sum_{s=0}^{n-1} \varphi(n - s - 1)|x(s)| - |x(n)|
\]
\[ A(n) |x(n)| + \sum_{s=0}^{n-1} |B(n, s)| \gamma(s)|x(s)| + |B(n, n)| \gamma(n)|x(n)| + \lambda_1(n) + \lambda_2(n)|x(n)| + \varphi(0)|x(n)| + \sum_{s=0}^{n-1} \varphi(n-s)|x(s)| - \sum_{s=0}^{n-1} \varphi(n-s-1)|x(s)| - |x(n)| \]

\[ = \left[ |A(n)| + |B(n, n)| \gamma(n) + \lambda_2(n) + \varphi(0) - 1 \right] |x(n)| + \lambda_1(n) + \sum_{s=0}^{n-1} \left[ |B(n, s)| \gamma(s) + \Delta_s \varphi(n-s-1) \right] |x(s)| \]

\[ \leq -\alpha|x(n)| + \lambda_1(n) \]

\[ \leq -\alpha|x(n)| + M. \quad (2.2) \]

By mimicking the proof of Theorem 1 of [14] we arrive at,

\[ |x(n)| \leq |\varphi(0)| R + \frac{M}{\alpha}, \]

for some positive constant \( R \). Hence, all solutions \( x(n) \) of (1.1) are uniformly bounded. Also, by a similar argument as in [14] we obtain

\[ \lim_{n \to \infty} \sup |x(n)| \leq \frac{M}{\alpha}. \]

Next, we assume that \( \lambda_1(n) = 0 \) and show that the zero solution of (1.1) is uniformly asymptotically stable. From (2.2), we have that

\[ \Delta V(n, x(\cdot)) \leq -\alpha|x(n)|. \]

Let \( \rho = 1 + \sum_{s=0}^{\infty} \varphi(s) \). As \( \Delta V(n, x(\cdot)) \leq 0 \) it follows that

\[ |x(n, n_0, \phi)| \leq V(n, x(\cdot)) \leq V(n_0, \phi) \]

\[ \leq |\phi| + \sum_{s=0}^{n_0-1} \varphi(n_0-s-1)|\phi| \]

\[ \leq ||\phi|| \left[ 1 + \sum_{s=0}^{\infty} \varphi(s) \right] \]

\[ \leq \epsilon, \]

for \( ||\phi|| \leq \delta \) with \( \delta = \epsilon/\rho \). Hence the zero solution of (1.1) is uniformly stable. For the rest of the proof, we refer the reader to Theorem 4 of [3]. This completes the proof.
In the next theorem we display a different type of Lyapunov functional and obtain a new variation of parameters formula, which will be used to show that all solutions of (1.1) are uniformly ultimately bounded. Also, when $\lambda_1(n) = 0$ we show that the zero solution of (1.1) is exponentially stable and uniformly exponentially stable.

**Theorem 2.2.** If

$$|A(n)| + k \sum_{j=n+1}^{\infty} |B(j, n)|\gamma(n) + |B(n, n)|\gamma(n) + \lambda_2(n) \leq 1 - \alpha$$  \hspace{1cm} (2.3)

for some $\alpha \in (0, 1)$ and $k = 1 + \varepsilon, \varepsilon > 0,$

$$|B(n, s)| \geq \lambda \sum_{j=n}^{\infty} |B(j, s)|$$  \hspace{1cm} (2.4)

where $\lambda \geq \frac{k\alpha}{\varepsilon},$ $0 \leq s < n < \infty,$ and

$$\sum_{s=0}^{n_0-1} \sum_{j=n_0}^{\infty} |B(j, s)| < \infty,$$ \hspace{1cm} (2.5)

then solutions of (1.1) are bounded. In addition to conditions (2.3) and (2.4) and (2.5), if $\lambda_1(n) = 0$ then the zero solution of (1.1) is exponentially stable.

**Proof.** Define

$$V(n, x(\cdot)) = |x(n)| + k \sum_{s=0}^{n-1} \sum_{j=n}^{\infty} |B(j, s)||f(s, x(s))|.$$ \hspace{1cm} (2.6)

Then along solutions of (1.1), we have

$$\Delta V(n, x(\cdot)) = |x(n+1)| - |x(n)| + k \sum_{s=0}^{n} \sum_{j=n+1}^{\infty} |B(j, s)||f(s, x(s))|$$

$$- k \sum_{s=0}^{n-1} \sum_{j=n}^{\infty} |B(j, s)||f(s, x(s))|$$
\[
\begin{align*}
&= \left| A(n)x(n) + \sum_{s=0}^{n} B(n, s)f(s, x(s)) + g(n, x(n)) \right| \\
&\quad - |x(n)| + k \sum_{s=0}^{n} \left[ \sum_{j=n}^{\infty} |B(j, s)||f(s, x(s))| \right] \\
&\quad - |B(n, s)||f(s, x(s))| - k \sum_{s=0}^{n} \sum_{j=n}^{\infty} |B(j, s)||f(s, x(s))| \\
&\leq \left[ |A(n)| + \lambda_2(n) + |B(n, n)|\gamma(n) \right] \\
&\quad + k \sum_{j=n+1}^{\infty} |B(j, n)|\gamma(n) - 1 |x(n)| \\
&\quad + (1 - k) \sum_{s=0}^{n-1} |B(n, s)||f(s, x(s))| + \lambda_1(n) \\
&\leq - \alpha|x(n)| + M + (1 - k) \sum_{s=0}^{n-1} |B(n, s)||f(s, x(s))| \\
&\leq - \alpha|x(n)| - \epsilon \lambda \sum_{s=0}^{n-1} \sum_{j=n}^{\infty} |B(j, s)||f(s, x(s))| + M \\
&\leq - \alpha \left[ |x(n)| + k \sum_{s=0}^{n-1} \sum_{j=n}^{\infty} |B(j, s)||f(s, x(s))| \right] + M \\
&= - \alpha V(n, x(\cdot)) + M. \quad (2.7)
\end{align*}
\]

From (2.7), we arrive at a variation of parameters formula, see [9] and [10], in terms of \( V(n, x(\cdot)) \). That is,

\[
V(n, x(\cdot)) \leq (1 - \alpha)^{n-n_0} V(n_0, x(\cdot)) + M \sum_{s=n_0}^{n-1} (1 - \alpha)^{n-s-1}
\]

\[
= (1 - \alpha)^{n-n_0} V(n_0, x(\cdot)) + \frac{M}{\alpha} - \frac{M}{\alpha} (1 - \alpha)^{n-n_0}
\]

\[
\leq (1 - \alpha)^{n-n_0} V(n_0, x(\cdot)) + \frac{M}{\alpha}.
\]

By (2.6), we have

\[
|x(n)| \leq V(n, x(\cdot)).
\]
Therefore,
\[ |x(n)| \leq (1 - \alpha)^{n-n_0} V(n_0, x(\cdot)) + \frac{M}{\alpha}. \tag{2.8} \]

Or,
\[ |x(n)| \leq ||\phi|| \left[ 1 + k \sum_{s=0}^{n_0-1} \sum_{j=n_0}^{\infty} |B(j, s)| \right] + \frac{M}{\alpha}. \]

Hence, all solutions of (1.1) are bounded. Next, we show the zero solution of (1.1) is exponentially stable. Let \( \lambda_1(n) = 0 \). Then from (2.8), we arrive at
\[ |x(n)| \leq (1 - \alpha)^{n-n_0} ||\phi|| \left[ 1 + k \sum_{s=0}^{n_0-1} \sum_{j=n_0}^{\infty} |B(j, s)| \right]. \]

Hence, the zero solution of (1.1) is exponentially stable due to the fact \((1 - \alpha) \in (0, 1)\). This completes the proof.

In the next corollary we show that if we strengthen condition (2.5), then Theorem 2.2 will imply uniform ultimate boundedness and hence, uniform exponential stability.

**Corollary 2.3.** Suppose the conditions of Theorem 2.2 hold except that (2.5) is replaced by
\[ \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} |B(j, s)| < \infty. \]

Then all solutions of (1.1) are uniformly ultimately bounded. Moreover, if \( \lambda_1(n) = 0 \) then, the zero solution of (1.1) is uniformly exponentially stable.

**Proof.** Let \( V(n, x(\cdot)) \) be defined by (2.6) and \( \rho = 1 + k \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} |B(j, s)| \). Then from (2.8), we have
\[ |x(n)| \leq (1 - \alpha)^{n-n_0} V(n_0, x(\cdot)) + \frac{M}{\alpha} \]
\[ \leq (1 - \alpha)^{n-n_0} ||\phi|| \left[ 1 + k \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} |B(j, s)| \right] + \frac{M}{\alpha} \tag{2.9} \]
\[ \leq (1 - \alpha)^{n-n_0} ||\phi|| \rho + \frac{M}{\alpha}. \]

Let \( B_1 \) be a positive constant such that \( ||\phi|| \leq B_1 \), on \([0, n_0]\). Let \( B_2 \) be a positive constant such that \( B_2 < B_1 \rho \). Then, for
\[ n - n_0 > \frac{\ln(B_2/(B_1 \rho))}{\ln(1 - \alpha)}, \]
we have
\[ |x(n)| \leq B_2 + \frac{M}{\alpha} =: B. \]
Thus, all solutions are uniformly ultimately bounded. If \( \lambda_1(n) = 0 \), then from (2.9), we have
\[ |x(n)| \leq (1 - \alpha)^{n-n_0}||\phi||\left[1 + k \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} |B(j, s)|\right], \]
which implies that the zero solution is uniformly exponentially stable.

3 Scalar Case

In this section we consider the scalar Volterra discrete equation with nonlinear perturbation
\[ x(n+1) = a(n)x(n) + \sum_{s=0}^{n} b(n, s)x(s) + g(n, x(n)), \quad n \geq 0 \]  
(3.1)

where \( g(n, x(n)) \) is as defined before. In the next theorem we will display a similar Lyapunov functional as in Theorem 2.2 and obtain boundedness and stability results for (3.1). In the past and present literature on the stability of the zero solution of (3.1), see \[4, \] \[5, \] \[8, \] \[11, \] \[12, \] and \[13, \] it has always been assumed that \( |a(n)| < 1 \), a condition that is very restrictive. Instead, we ask that \( |a(n) + b(n, n)| < 1 \).

**Proposition 3.1.** Let \( Q(n) = a(n) - H(n, n) \) and \( \Delta_n H(n, s) = b(n, s) \). Then (3.1) is equivalent to
\[ x(n+1) = Q(n)x(n) + \Delta_n \left[ \sum_{s=0}^{n-1} H(n, s)x(s) \right] + g(n, x(n)). \]  
(3.2)

**Proof.** From (3.2) we have,
\[
x(n+1) = Q(n)x(n) + \sum_{s=0}^{n} H(n+1, s)x(s) - \sum_{s=0}^{n-1} H(n, s)x(s) + g(n, x(n))
\]
\[
= Q(n)x(n) + \sum_{s=0}^{n} H(n+1, s)x(s) - \sum_{s=0}^{n} H(n, s)x(s) + H(n, n)x(n) + g(n, x(n))
\]
\[
= a(n)x(n) + \sum_{s=0}^{n} \Delta_n H(n, s)x(s) + g(n, x(n))
\]
\[
= a(n)x(n) + \sum_{s=0}^{n} b(n, s)x(s) + g(n, x(n)).
\]
This completes the proof of the proposition. The next result depends on the results obtained by Raffoul in [15]. For completeness, we state the following theorem. Consider the nonlinear discrete system

\[ x(n + 1) = G(n, x(s); 0 \leq s \leq n) \overset{def}{=} G(n, x(\cdot)) \]  

(3.3)

where \( G : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}^k \) is continuous in \( x \).

**Theorem 3.2 [15]**. Let \( \varphi(n, s) \) be a scalar sequence for \( 0 \leq s \leq n < \infty \) and suppose that \( \varphi(n, s) \geq 0, \Delta_n \varphi(n, s) \leq 0, \Delta_s \varphi(n, s) \geq 0 \) and there are constants \( B \) and \( J \) such that \( \sum_{n=0}^{\infty} \varphi(n, s) \leq B \) and \( \varphi(0, s) \leq J \). Also, suppose that for each \( n_0 \geq 0 \) and each bounded initial function \( \phi : [0, n_0] \to \mathbb{R}^k \), every solution \( x(n) = x(n, n_0, \phi) \) of (3.3) satisfies

\[ W_1(|x(n)|) \leq V(n, x(\cdot)) \leq W_2(|x(n)|) + \sum_{s=0}^{n-1} \varphi(n, s)W_3(|x(s)|) \]

and

\[ \Delta V_{(3.3)}(n, x(\cdot)) \leq -\rho W_3(|x(n)|) + K \]

for some constants \( \rho \) and \( K \geq 0 \), where the \( W_i : [0, \infty) \to [0, \infty) \) are continuous with \( W_i(0) = 0, W_i(r) \) strictly increasing, and \( W_i(r) \to \infty \) as \( r \to \infty, i = 1, 2, 3 \). Then solutions of (3.3) are uniformly bounded.

**Theorem 3.3.** Suppose

\[ \Delta_s |H(n, s)| \geq 0, \]

where

\[ \Delta_n H(n, s) = b(n, s). \]

If

\[ \Delta_n |H(n, s)| - P|H(n, s)| \leq 0, \quad n \geq 1, \]

\[ \sum_{s=0}^{\infty} \sum_{u=0}^{\infty} |H(u, s)| < \infty, \]

and

\[ |a(n) + b(n, n)| + \lambda_2(n) + P \sum_{u=n+1}^{\infty} |H(u, n)| \leq 1 - \alpha, \quad \alpha \in (0, 1), \quad n \geq 0 \]

(3.5)

for some positive constant \( P \), then solutions of (3.1) are uniformly bounded.
**Proof.** Define the Lyapunov functional \( V(n, x(\cdot)) \) by

\[
V(n, x(\cdot)) = |x(n)| + P \sum_{s=0}^{n-1} \sum_{u=n}^{\infty} |H(u, s)||x(s)|.
\]

Then along solutions of (3.2) we have,

\[
\Delta V(n, x(\cdot)) = |x(n+1)| - |x(n)| + P \sum_{u=n+1}^{\infty} |H(u, n)||x(n)|
- P \sum_{s=0}^{n-1} |H(n, s)||x(s)|
+ P \sum_{u=n+1}^{\infty} |H(u, n)||x(n)| - P \sum_{s=0}^{n-1} |H(n, s)||x(s)|
\]

\[
= |Q(n)x(n) + \Delta_n \left[ \sum_{s=0}^{n-1} H(n, s)x(s) \right] + g(n, x(n))| - |x(n)|
+ P \sum_{u=n+1}^{\infty} |H(u, n)||x(n)| - P \sum_{s=0}^{n-1} |H(n, s)||x(s)|
\]

\[
\leq \left( |Q(n) + H(n + 1, n)| + \lambda_2(n) + P \sum_{u=n+1}^{\infty} |H(u, n)| - 1 \right)|x(n)|
+ P \sum_{s=0}^{n-1} \left[ |\Delta_n H(n, s)| - P|H(n, s)| \right]|x(s)| + M.
\]

Note that since \( \Delta_n H(n, s) = b(n, s) \) we have that,

\[
H(n, s) = \sum_{u=0}^{n-1} b(u, s),
\]

which implies that

\[
Q(n) + H(n + 1, n) = a(n) - H(n, n) + H(n + 1, n)
= a(n) + \sum_{u=0}^{n} b(u, n) - \sum_{u=0}^{n-1} b(u, n)
= a(n) + b(n, n).
\]
Thus,
\[
\Delta V(n, x(\cdot)) \leq \left( |a(n) + b(n, n)| + \lambda_2(n) + P \sum_{u=n+1}^{\infty} |H(u, n)| - 1 \right) |x(n)| + M
\]
\[
\leq -\alpha|x(n)| + M.
\]

Let \( \varphi(n, s) = \sum_{u=n}^{\infty} |H(u, s)|. \) Then all the conditions of Theorem 3.2 [15] are satisfied, and hence solutions of (3.1) are uniformly bounded. This completes the proof.

We end the paper with a theorem about the stability of the zero solution of (3.1).

**Theorem 3.4.** Suppose (3.4) and (3.5) hold. If
\[
\sum_{u=0}^{\infty} \sum_{s=0}^{\infty} |H(u, s)| < \infty,
\]
and \( \lambda_1(n) = 0, \) then the zero solution of (3.1) is uniformly asymptotically stable.

**Proof.** Let \( V(n, x(\cdot)) \) be defined as in Theorem 3.3. Then, along solutions of (3.1) we have
\[
\Delta V(n, x(\cdot)) \leq -\alpha|x(n)|.
\]
The result follows from Theorem 4 of [3]. This completes the proof.

We remark that, the above theorem shows that the zero solution of (3.1) is uniformly asymptotically stable without requiring \( |a(n)| < 1 \) provided that \( |a(n) + b(n, n)| < 1. \) Thus, the condition \( |a(n) + b(n, n)| < 1 \) is a major improvement over the results in the literature [3], [5], [6] and [11] concerning uniform asymptotic stability of the zero solution of discrete equations that are similar to (3.1), in case \( a(n) \) and \( b(n, n) \) are of opposite signs.

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**References**


