A Probability Primer

A random walk down a probabilistic path leading to some stochastic thoughts on chance events and uncertain outcomes.

Are you holding all the cards??
Random Events

♦ A random event, E, will occur with some probability denoted by P(E) where $0 < P(E) < 1$.
♦ $P(E) = 0$ describes an impossible event
♦ while $P(E) = 1$ denotes a certain event.

The collection of all possible outcomes (events) relative to a random process is called the sample space, S where $S = \{E_1, E_2 \ldots E_k\}$ and $P(S) = 1$. 
Probability review

Complementary Events, Unions and Intersections of Events

Let $A =$ the event, the failure of component 1,
$B =$ the event, the failure of component 2.

Then $A \cap B =$ the event, both components have failed,

$A^c \cap B =$ the event, component 1 did not fail and component 2 failed,

$A^c \cup B =$ the event, component 1 did not fail or component 2 failed.
Probability of Complementary Events, Unions and Intersections of Events

\[ P(A^c) = 1 - P(A) \]

\[ P(A \cup B) = P(A) + P(B) \text{ if } A \text{ and } B \text{ are mutually exclusive} \]

\[ P(A \cap B) = 0 \text{ if } A \text{ and } B \text{ are mutually exclusive} \]
Independent Events

Two events A and B are **independent** if and only if

\[ P(A \cap B) = P(A) \times P(B) \]

**Example:** Let A = failure of component 1 and B = failure of component 2; where

\[ P(A) = 0.1 \text{ and } P(B) = 0.2 \]

Then

\[ P(A \cap B) = (0.1) \times (0.2) = 0.02 \]

is the probability both components fail.
Conditional Probability

If two events are **dependent**, then the occurrence of one event changes the probability of the other event.

Define the conditional probability:

\[
P(A|B) = \frac{P(A \cap B)}{P(B)}
\]

Then

\[
P(A \cap B) = P(A|B) \cdot P(B)
\]
Conditional Probability
Example - load sharing system

Two components share a common load.
If one component fails, the probability
the other component will fail increases.
Let $A =$ the event, component 1 fails
$B =$ the event, component 2 fails

Given $P(A) = P(B) = .10$ and $P(A|B) = P(B|A) = .90$

Then $P(A \cap B) = P(A) \cdot P(B|A) = .10 \cdot (.90) = .09$
Addition Formula

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
\[ = P(A) + P(B) - P(A|B) \cdot P(B) \]
\[ = P(A) = P(B) - P(A) \cdot P(B) \quad \text{if independent} \]
Addition Formula Example

Example: Let $A = \text{failure of component 1}$ and $B = \text{failure of component 2}$; where $P(A) = .1$ and $P(B) = .2$

Then the probability at least one component fails is given by:

$$P(A \cup B) = .1 + .2 - (.1) (.2) = .28$$

assuming independence
Random Variables

☐ A random variable is a variable which takes on numerical values in accordance with some probability distribution.

☐ Random variables may be either continuous (taking on real numbers) or discrete (usually taking on non-negative integer values).

☐ The probability distribution which assigns probabilities to each value of a discrete random variable, or assigns a probability over an interval of values of a continuous random variable, can be described in terms of a probability mass function (PMF), \( p(x) \) in the discrete case, and a probability density function (PDF), \( f(x) \), in the continuous case.

☐ For both discrete and continuous distributions, a cumulative distribution function (CDF), \( F(x) \) is defined where \( P\{X < x\} = F(x) \).

☐ By convention, capital letters represent the random variable while the corresponding small letters denote particular values the random variable may assume.
Random Variables - Examples

★ Let $T$ = a continuous random variable, the time to failure of a component,
★ $Y$ = a discrete random variable, the number of failures occurring in some time $t$,
★ $W$ = a continuous random variable, the time to repair a failed system, and
★ $X$ = a discrete random variable, the number of cycles until the first failure occurs.
Discrete Distributions
Cumulative Distribution Function (CDF)

\[ F(x) = \Pr\{X \leq x\} = \sum_{\xi \text{ all}} p(\xi) \]

where \( p(x) = \Pr\{X=x\} \)
Discrete Distributions

\[ \sum_{\text{all } x} p(x) = 1 \]

\[ \mu = \sum_{\text{all } x} xp(x) \]

\[ \sigma^2 = \sum_{\text{all } x} (x - \mu)^2 p(x) \]

variance
Binomial Distribution

X = a discrete random variable, the number of “successes” from among \( n \) independent trials having a constant probability of success equal to \( p \). \( X = 0, 1, 2, \ldots, n \)

\[
p(x) = \binom{n}{x} p^x (1 - p)^{n-x}
\]

\( p^x \) = prob of \( x \) successes \hspace{1cm} (1-p)^{n-x} = \text{prob of } n-x \text{ non-successes}

\[
\binom{n}{x} = \frac{n!}{x!(n-x)!} = \text{number of ways of achieving } x \text{ successes}
\]
Binomial Distribution

Example

Let $X$ = a discrete random variable, the number of failed components among 5 independent and identical components where each component has one chance in 100 of failing.

$$p(x) = \binom{5}{x} .01^{x} (.99)^{5-x}$$

$$E(X) = np = 5 (.01) = .05 \text{ and}$$
$$Var(x) = np(1-p) = 5 (.01) (.99) = .0495$$

$$Pr\{X\leq1\} = p(0) + p(1) = .99^5 + 5(.01)(.99^4) = .999$$
Poisson Distribution

$X = a$ discrete random variable, the number of random occurrences (events) in a specified time. $X = 0, 1, 2, \ldots$

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\mu = E(X) = Var(X) = \lambda$$
Poisson Distribution

Let \( X \) = a discrete random variable, the number of failures and subsequent repairs of a restorable system over a one year period. Assuming \( X \) has a Poison distribution with a mean of 2 failures per year, then the probability of no more than one failure a year is

\[
P \{ X \leq 1 \} = F(1) = \sum_{x=0}^{1} \frac{e^{-2}2^x}{x!} = .406
\]
Continuous Distributions

1. \( 0 < F(x) < 1 \)

2. \( P\{X < x\} = F(x) = \int_{-\infty}^{x} f(\xi)d\xi \)

3. \( \int_{-\infty}^{\infty} f(x)dx = 1 \)

4. \( P\{a < X < b\} = \int_{a}^{b} f(x)dx = F(b) - F(a) \)

5. \( \mu = \int_{-\infty}^{\infty} x f(x)dx \)