21. Viewed from the side, this looks like

Let \( y \) be the \( y \)-coordinate of the base of the cone. Let \( x \) be the \( x \)-coordinate of the edge of the cone's base. Then \( a \) is the height of the cone, and \( x \) is its radius.

Since \( (x, y) \) is a point on the sphere, we know that \( x^2 + y^2 = a^2 \) or \( x^2 = a^2 - y^2 \). The volume of the cone is

\[
V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} a^2 (a + y) = \frac{\pi}{3} (a^2 - y^2)(a + y), \quad -a \leq y \leq a.
\]

\[
V = \frac{\pi}{3} (a^2 + a^2 y - a y^2 - y^3)
\]

\[
V' = \frac{\pi}{3} (a^2 - 2ay - 3y^2) = -\frac{\pi}{3} (3y - a)(y + a)
\]

So \( V'(y) = 0 \) when \( y = -a, \frac{a}{3} \).

\[
V(-a) = 0
\]

\[
V\left(\frac{a}{3}\right) = \frac{\pi}{3} \left(a^2 - \frac{a^2}{a}\right)\left(a + \frac{a}{3}\right) = \frac{\pi}{3} \left(\frac{8a^2}{9}\right) \left(\frac{4a}{3}\right) = \frac{32\pi a^3}{81}
\]

\[
V(a) = 0
\]

The largest volume is \( \frac{32\pi a^3}{81} \).