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AN ANALYTICAL SOLUTION FOR DYNAMIC THERMAL TRANSMISSION LOADS

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Abstract

Thermal mass in building envelopes reduces the magnitude of diurnal conductive loads and, in some cases, reduces energy use for heating and cooling. In general, dynamic simulation is required to estimate the magnitude of these effects. This paper seeks to derive closed-form expressions of thermal time lag and amplitude dampening through building envelope structures, to aid in the design process before detailed simulation is performed, and to improve intuition about the effects of thermal mass in buildings.

An analytical solution for the temperature distribution through an infinite wall subjected to a sinusoidal temperature boundary condition is derived. The solution is verified by comparison with a finite element solution. Next, it is hypothesized that the analytical solution for an infinite wall could also describe the temperature variation on the inside surface of a finite wall. The hypothesis is tested by comparing the temperature distribution predicted by the analytical solution to the temperature distribution predicted by a finite element model of a finite wall. The results confirm that analytical solution adequately described the temperature variation in a finite wall. Based on the analytical solution, separate closed-form expressions of time delay and amplitude dampening for thermal load transmission through building envelopes are derived. The use of these expressions is demonstrated for light frame and concrete walls.

Introduction

Thermal mass has been used in building construction since ancient times. Today, an in-depth understanding of thermal mass can benefit the modern quest for sustainable buildings. Thermal mass acts as a buffer by absorbing heat during the day when the surroundings are hot and releasing heat at night when the surroundings are cooler. The quantity of heat absorbed and released depends on the thermal mass and diurnal drivers such as outdoor air temperature swing, solar radiation and internal loads. In general, thermal mass reduces and shifts peak loads, which moderates the indoor temperature. For example, Figure 1 shows commercial building cooling loads delayed until after an office closes and buildings electrical demand has decreased.

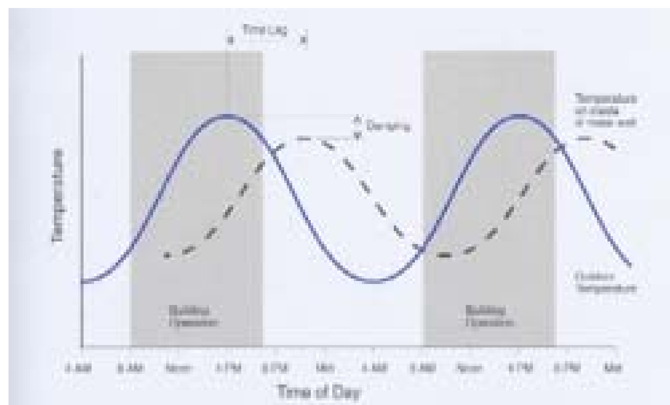

Figure 1. Damping and lag effect of thermal mass [1].

The current method of calculating time-dependent heating and cooling loads recommended by ASHRAE is called the heat balance method. According to Petersen et al. [2], “There are probably more ways to formulate the wall conduction process than any of the other processes. As a result it is the topic that has received the most attention over the years.” Petersen et al. list numerical finite difference, numerical finite element, transform methods and time series methods as possible ways to model the process. The method for calculating wall conduction used in the ASHRAE heat balance method is called the computational transfer function procedure (CTF) [3].

Other research about the use of thermal mass in walls to has utilized experimental and simulation methodologies. For example, Kosny, J. et al. used finite difference modeling of a wall assembly and performed experiments to gauge thermal mass performance [4]. Colliver et al. experimentally investigated the placement of thermal mass in the wall and its impact on the energy use [5]. Kissock et al. used finite difference modeling to investigate thermal mass effects from phase change materials [6, 7, 8].

In this paper, a closed-form relationship is developed for estimating time delay and amplitude dampening of heat loads through walls, and validated using finite element methods. The closed-form solution developed here allows the reduction and delay in peak transmission loads to be explicitly calculated, without the need for time-consuming simulation. Hence, it presents a compliment to the building energy simulation approach. The method may be especially useful for comparing the thermal response of various wall materials at early stages in the design process before the design is developed enough to allow simulation.

Analytical Solution

Assuming one-dimensional heat transfer, the governing equation for unsteady-state unidirectional heat conduction is:

$$\alpha \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u}{\partial t} \qquad (1)$$

where α = thermal diffusivity,
u(X(x),t)=Excess temperature
$= T(X(x),t) - T_{environmen\ t}$
$u_0 = T_0$

For an infinite wall, the problem is defined as in Figure 2.

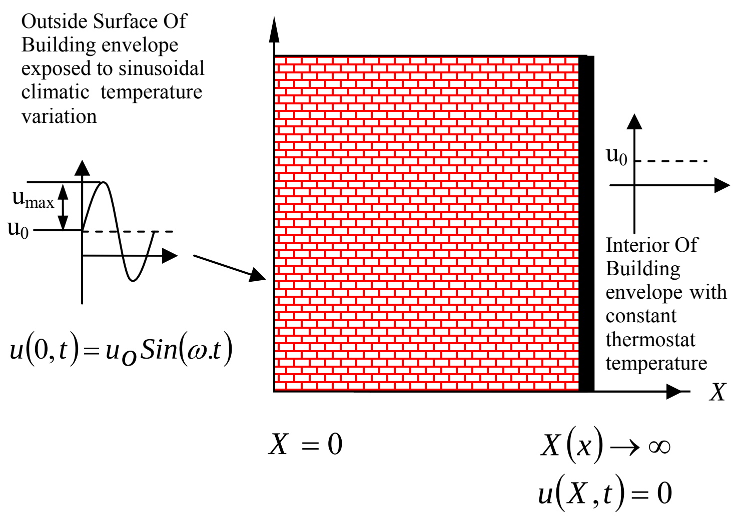

Figure 2. Problem definition for infinite wall.

The first boundary condition for this problem is a sinusoidal temperature at the outside surface of the wall, given by:

$u_{o,t} = u_{\max} Sin(\omega \cdot t)$ at X (x)=0 for t ≥ 0 (2)

The wall is assumed to be infinitely thick, and there is no temperature change at the end of the wall. Therefore, as x tends to infinity, $u(X,t)$ tends to zero.

Since the boundary condition is sinusoidal in temperature, we assume the solution takes the form of an imaginary exponential function in time and a real function in distance,

$$u(X,t) = X(x)\left[ie^{i\omega\cdot t}\right] \qquad (3)$$

Substituting Equation 3 into Equation 1 gives:

$$\alpha \frac{\partial^2}{\partial x^2}\left[Xie^{i\omega\cdot t}\right] = \frac{\partial}{\partial t}\left[Xie^{i\omega\cdot t}\right]$$

$$\alpha i \frac{d^2 X}{dx^2} + X\omega = 0 \Rightarrow \frac{d^2 X}{dx^2} - \frac{X\omega.i}{\alpha} = 0 \qquad (4)$$

Solution of Equation 4 is given by:

$$X(x) = Ae^{x\sqrt{i}\sqrt{\omega/\alpha}} + Be^{-x\sqrt{i}\sqrt{\omega/\alpha}} \qquad (4b)$$

From Demoivre's theorem (Equation 4c), where

$n = 1/2\ and\theta = \pi/2$:

$$Cos\theta + iSin\theta = Cos(n\theta) + iSin(n\theta) \qquad (4c)$$

The square root of i is:

$$\sqrt{i} = \left(1 + i/\sqrt{2}\right) \qquad (4d)$$

Substituting Equation 4d into Equation 4b gives:

$$X(x) = Ae^{x.\sqrt{(1+i)/\sqrt{2}}.\sqrt{\omega/\alpha}} + Be^{-x.\sqrt{(1+i)/\sqrt{2}}.\sqrt{\omega/\alpha}} \qquad (4e)$$

From Equation (3), the temperature variation at a distance x into the wall and time t is given by:

$$u(X(x),t) = Ae^{i\left[\omega.t + x\sqrt{\omega/2\alpha}\right]} i.e^{x\sqrt{\omega/2\alpha}} + e^{-i\left[x\sqrt{\omega/2\alpha} - \omega t\right]} i.e^{-x\sqrt{\omega/2\alpha}}$$

Using the secondary boundary condition, $u(\infty,t) = 0$ as $X \to \infty$, causes A to be zero. Then,

$\therefore A = 0$

$$\therefore u(X(x),t) = Bie^{-i\left[x\sqrt{\omega/2\alpha} - \omega t\right]}.e^{-x\sqrt{\omega/2\alpha}} \qquad (5)$$

Substituting the boundary conditions that at $x = 0$ then $u(0,t) = u_0 Sin(\omega t)$ into Equation 5, gives:

$$u_0 Sin(\omega t) = BiCos(\omega t) - BiSin(\omega t)$$

Equating the real parts on both sides of the equation gives:

$$B = u_0$$

Substituting B into Equation 5 gives:

$$u(X(x),t) = u_0 .i.e^{-x\sqrt{\omega/2\alpha}}\left[Cos\left(x\sqrt{\frac{\omega}{2\alpha}} - \omega t\right) + i.Sin\left(x\sqrt{\frac{\omega}{2\alpha}} - \omega t\right)\right]$$

$$u(X(x),t) = u_0 e^{-x\sqrt{\omega/2\alpha}}\left[Sin.\left(x\sqrt{\frac{\omega}{2\alpha}} + \omega t\right)\right] + imaginary \quad (6)$$

Since $u(X,t)$'s imaginary part is 90^0 out of phase with the real part and constitutes the same solution as the real part, we can neglect the imaginary part of Equation 6 to get:

$$u(X(x),t) = u_O e^{-x\sqrt{\omega/2\alpha}}\, Sin\left(-x.\sqrt{\frac{\omega}{2\alpha}} + \omega t\right) \quad (7)$$

The one dimensional special transient temperature variation across the wall cross section is then:

$$T(X(x),t) = T_{environ} + T_O e^{-x\sqrt{\omega/2\alpha}}\, Sin\left(-x.\sqrt{\frac{\omega}{2\alpha}} + \omega.t\right) \quad (8)$$

A graph of the temperature variation at a single instant as a function of distance into a wall responding to a sinusoidal temperature boundary condition as predicted by Equation 8 is shown in (Figure 3).

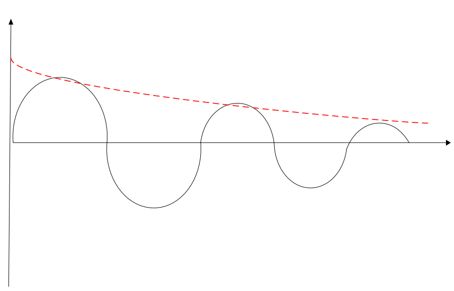

Figure 3. Temperature variation as function of distance into all with sinusoidal temperature as boundary condition at x = 0.

Amplitude Dampening

The amplitude of the temperature at any point inside the wall dampens with the depth into the wall. Amplitude at depth x will be maximum when:

$$Sin\left(-x\sqrt{\frac{\omega}{2\alpha}} + \omega t\right) = 1;$$

$$\Rightarrow -x\sqrt{\frac{\omega}{2\alpha}} + \omega t = \frac{n\pi}{2}$$

Thus, the excess temperature variation is:

$$u(X) = u_O e^{-x\sqrt{\omega/2\alpha}} \quad (9)$$

And the temperature inside the wall can be written as:

$$T(X(x)) = T_{environment} + T_O e^{-x\sqrt{\omega/2\alpha}} \quad (10)$$

Equation 10 shows that the amplitude decreases exponentially with increasing x. This means that the amplitude of the temperature swing becomes negligible at a relatively small depth inside the wall thickness, and if the thickness of the wall is greater than this depth, there is no temperature variation along the 'x' direction. The amplitude also varies with square root of the inverse of the thermal diffusivity of the wall material. This means that the amplitude of the temperature becomes larger when the thermal diffusivity increases. Thus, the amplitude of the temperature swing becomes larger in light-weight walls.

Time Lag

The time lag at a distance into the wall is the difference between the peaks of imposed sinusoid at the left end and the temperature variation at same distance into the wall. As before, the amplitude at depth x will be maximum when:

$$Sin\left(-x\sqrt{\frac{\omega}{2\alpha}} + \omega t\right) = 1$$

$$\Rightarrow \omega t - x\sqrt{\frac{\omega}{2\alpha}} = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\ldots\ldots\ \ldots\ldots\ldots \frac{(2n+1)\pi}{2} \quad (11)$$

$$\Rightarrow \omega t - x\sqrt{\frac{\omega}{2\alpha}} = \frac{(2n+1)\pi}{2}$$

From the Equation 11, the time lag is:

$$\Delta t = \sqrt{\frac{1}{2\omega\alpha}}x \quad (12)$$

Equation 12 shows that the time lag increases with x and decreases with increasing ω and α.

Finite Element Solution

To verify the analytical solutions developed above, a finite element analysis was performed on the cross section of wall with the same boundary conditions as in the analytical solution (Figure 4). A rectangular block with height of 0.1 meters and cross-sectional length of 1m is considered. To simulate a wall of infinite thickness, the right end of the block is meshed with a thermal line element. The block is divided into ten blocks of 0.1 m each along the length and five divisions along the height. The mesh is refined more at the left, to get a finer resolution of the temperature variation near that end.

Figure 4. ANSYS® Finite element model of an infinite wall.

A finite wall of cross-sectional length 0.2m and height 0.1m was also modeled (Figure 5). Five elements were considered along the height and twenty elements along the length for resolving the temperature variation along those dimensions.

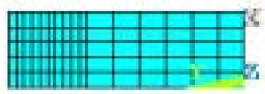

Figure 5. ANSYS® finite element model of a finite wall.

The right edge of the finite element model is the interior of the building envelope and a fixed temperature of 50^0C is imposed on this edge. A sinusoidal temperature variation was imposed at the left end of both the infinite and finite walls. Temperatures within the wall were simulated using the thermal module of ANSYS® [9]. The temperature profiles are compared against the derived analytical transient temperature profiles at a distance of 0.01m and 0.08m into the wall from the left end (Figures 6 and 7). The figures show that the results from the finite, infinite and analytical solutions are nearly identical at both depths. This validates that the analytical solution can be applied to finite walls, even though it was derived for the case of an infinite wall.

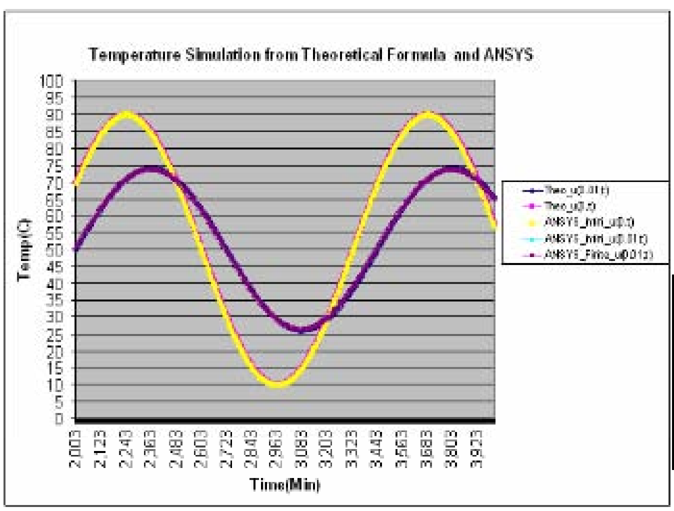

Figure 6. Comparison of analytical and finite element results for an infinite and finite wall at 0.01 meters.

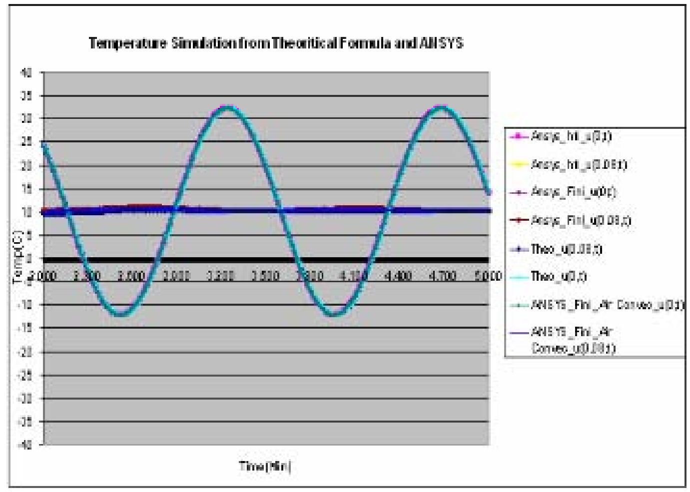

Figure 7. Comparison of analytical and finite element results for an infinite and finite wall at 0.08 meters.

The amplitude change and the time delay were also calculated using the analytical formulas and the ANSYS simulation. Tables 1 show that the analytical and simulation results are practically identical.

Table 1. Comparison of ANSYS® and analytical solutions.

Comparison of ANSYS® and analytical values at x = 0.01 m		
	Time delay(Min)	Amplitude change(C)
ANSYS®	113.3	15.7895
Theoretical	113.3	15.7895
Comparison of ANSYS and analytical values at x = 0.08 m		
	Time delay(Min)	Amplitude change(C)
ANSYS®	880	38.4656
Theoretical	882	38.4656

Example Applications

Based on these results, Equations 9 and 12 for time lag and amplitude dampening may be useful during the design process for estimating thermal load transmission effects without the need for detailed simulation. Consider, for example, the application of these equations to two walls, one made of wood and one of concrete. The properties of the wood and concrete walls are shown in the Table 2.

Table 2. Properties of wood and concrete walls.

Material	Thermal Conductivity, K (W/mK)	Density, ρ (kg/m^3)	Specific Heat, Cp (kJ/kg K)
Wood	0.1154	512.6	1.382
Concrete	1.731	2243	0.9211

The amplitude dampening, from Equation (9) can be rewritten as a function of the density, thermal conductivity, specific heat and the frequency of the imposed sinusoid, ω by substituting $\frac{\bar{k}}{\bar{\rho}\bar{C}_p}$ for the thermal diffusivity.

$$u(X) = u_o e^{-x\sqrt{\omega \bar{\rho} \bar{C}_p / 2\bar{k}}} \qquad (13)$$

The ratio of the amplitude at a distance x and the maximum amplitude is given by:

$$\left(\frac{u(x)}{u_0}\right) = e^{-x\sqrt{\frac{\omega\rho\ Cp}{2k}}} \qquad (14)$$

Wall thickness is plotted against the amplitude ratios of each type of wall in Figures 8. The figure show reducing the magnitude of the amplitude of the temperature variation on the inside surface by 50% would require a wood wall to be about 0.05 meters thick, whereas the concrete wall would need to be about 0.10 m thick.

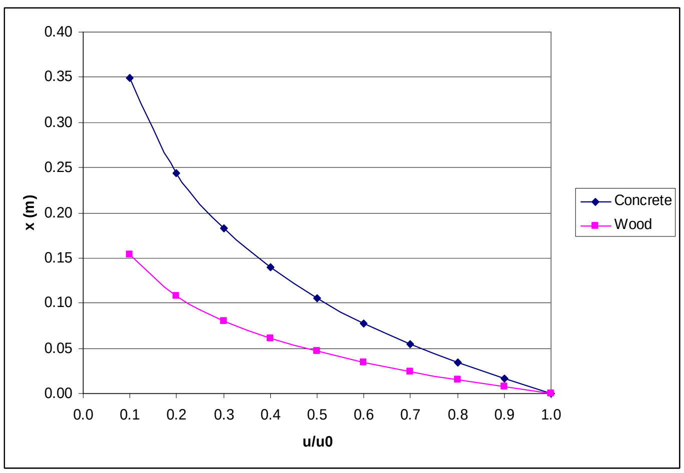

Figure 8. Amplitude dampening for wood and concrete walls

This results shows that thermal mass (m cp) alone is not the only determinant of thermal mass effects. The fact that the conductivity of wood is more than an order of magnitude less than the conductivity of concrete has a major effect on the amplitude of the thermal load.

Similarly Equation (12) for time delay can be written as:

$$\Delta t = x\sqrt{\frac{\bar{\rho} C_p}{2\omega \bar{k}}} \qquad (15)$$

Using Equation 15, the thicknesses required to achieve a given time delay for wood and concrete walls are shown in Figure 9. In this case, wall thickness and time delay vary linearly. As in Figure 8, the wall thickness required to increase the time delay of a diurnal thermal load is less for wood than concrete.

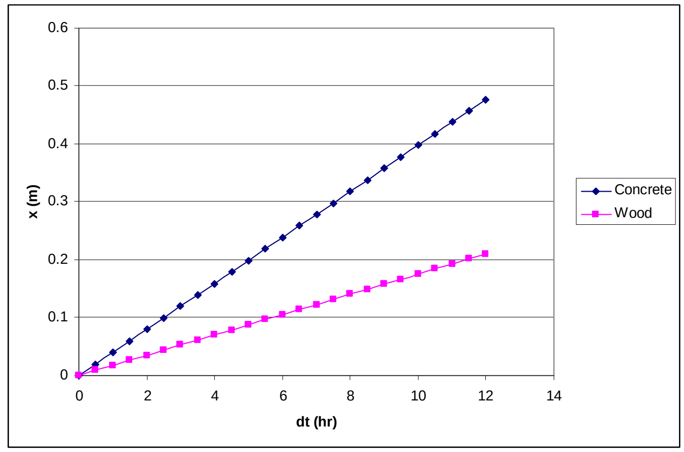

Figure 9. Time delay profiles for the light frame wall and concrete

Figures 8 and 9 conclude that the wood wall has more amplitude dampening and lesser time delay than the concrete. The boundary condition imposed on the wall is purely analytical for the sake of simplicity of the analysis. The actual diurnal temperature is not sinusoidal. Moreover, the wall temperature is dependent upon the surrounding air temperature. A convective heat transfer condition should also be specified on the wall for actual heat transfer analysis.

Summary

An analytical solution for the temperature distribution through an infinite wall subjected to a sinusoidal temperature boundary condition was derived. The solution was verified by comparison with a finite element solution. Next, it was hypothesized that the analytical solution for an infinite wall could also describe the temperature variation on the inside surface of a finite wall. The hypothesis was tested by comparing the temperature distribution predicted by the analytical solution to the temperature distribution predicted by a finite element model of a finite wall. The results confirmed that analytical solution adequately described the temperature variation in a finite wall.

Based on the analytical solution, separate closed-form expressions of time delay and amplitude dampening for thermal load transmission through building envelopes were derived. These expressions may be useful during the design process for estimating thermal effects of various wall structures. The use of these expressions was demonstrated for light frame and concrete walls.

This work is limited to homogeneous walls and sinusoidal boundary conditions. Future work will explore the applicability of these expressions to non-homogeneous walls. Future work will also seek to develop specific guidelines for estimating peak load reduction and energy savings from thermal mass in building envelopes as a function of materials and climate.

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